Finite Element Analysis with “AMORE”

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“AMORE” means “Love”

The AMORE paradigm –

Two ingredients --

-- easier meshing, hence less engineering effort required

-- more accurate solutions

The result is that a finite element analysis can be much more effective in two-dimensional (2D) and three-dimensional (3D) analyses

AMORE

Automatic **M**eshing with **O**verlapping and **R**egular **E**lements

-- Scheme is for solids, CAD defined but not restricted to CAD functions – The steps are:

-- Analysis part is immersed in a “Cartesian grid”

-- Boundary of analysis part is discretized, hence a 3D geometry requires a 2D meshing of the boundary (simple 3-node triangles are used). The mesh is only used to describe the geometry.

-- Cells of Cartesian grid inside the geometry are turned into undistorted regular effective finite elements, e. g. elements with incompatible modes are used. All other cells are removed.

-- “Overlapping elements” are used to fill in the empty space, since the OFE are distortion -insensitive, distorted and even sliver elements can be employed.

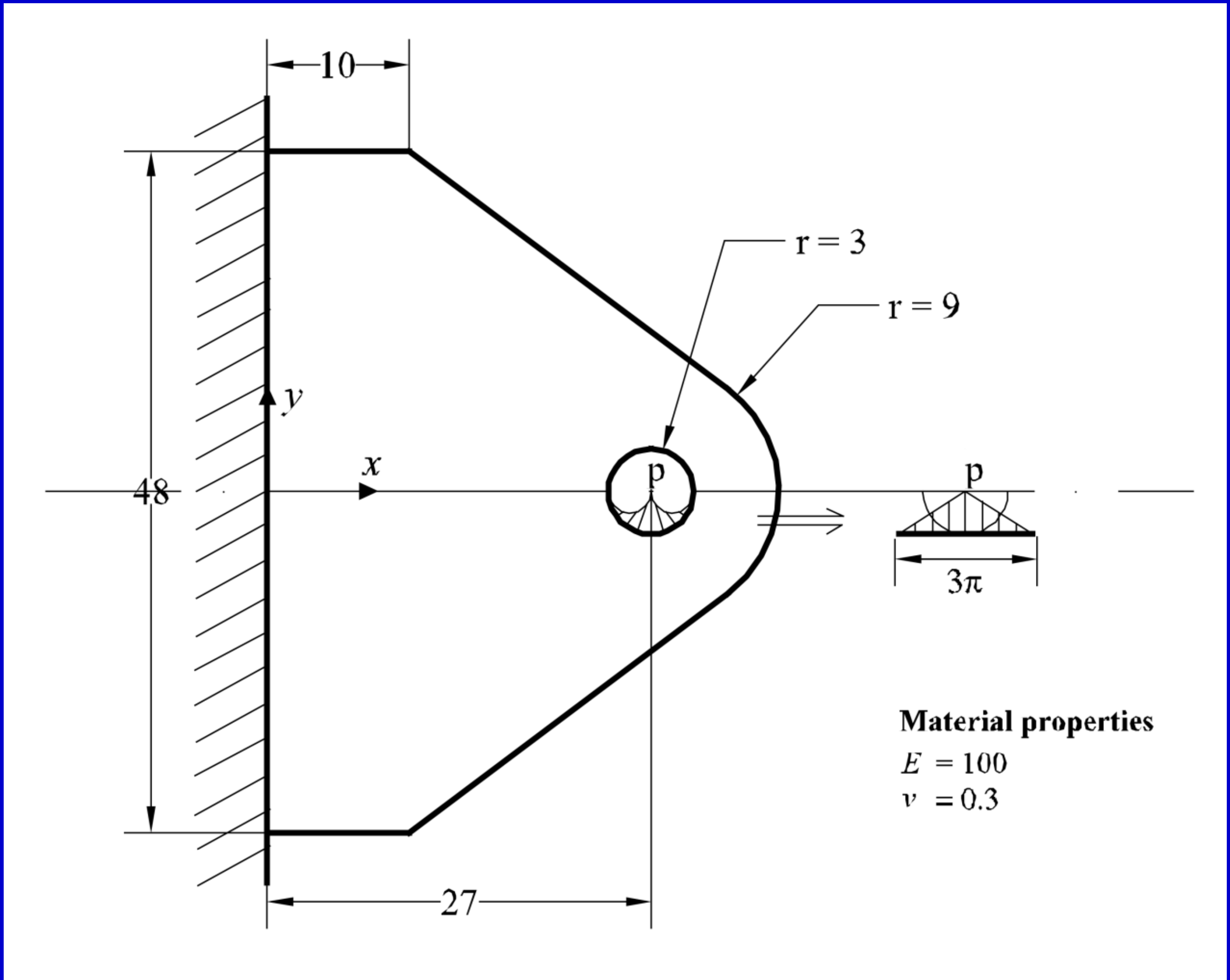
The premise is: “The meshing is easier, the solution effort is less and the complete analysis is more reliable.”

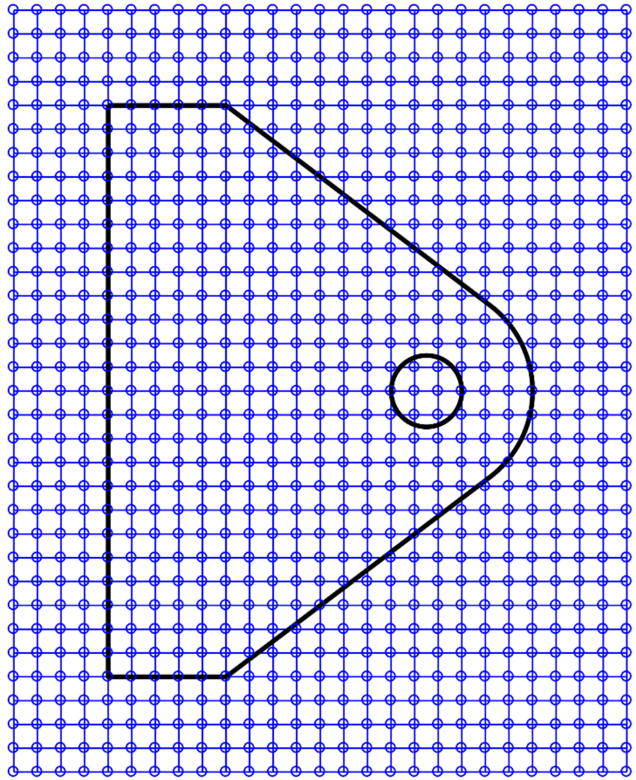
AMORE depends on effective overlapping finite elements (OFE) --

We need to discuss the formulation of the elements, give fundamental observations & show demonstrative solutions in --

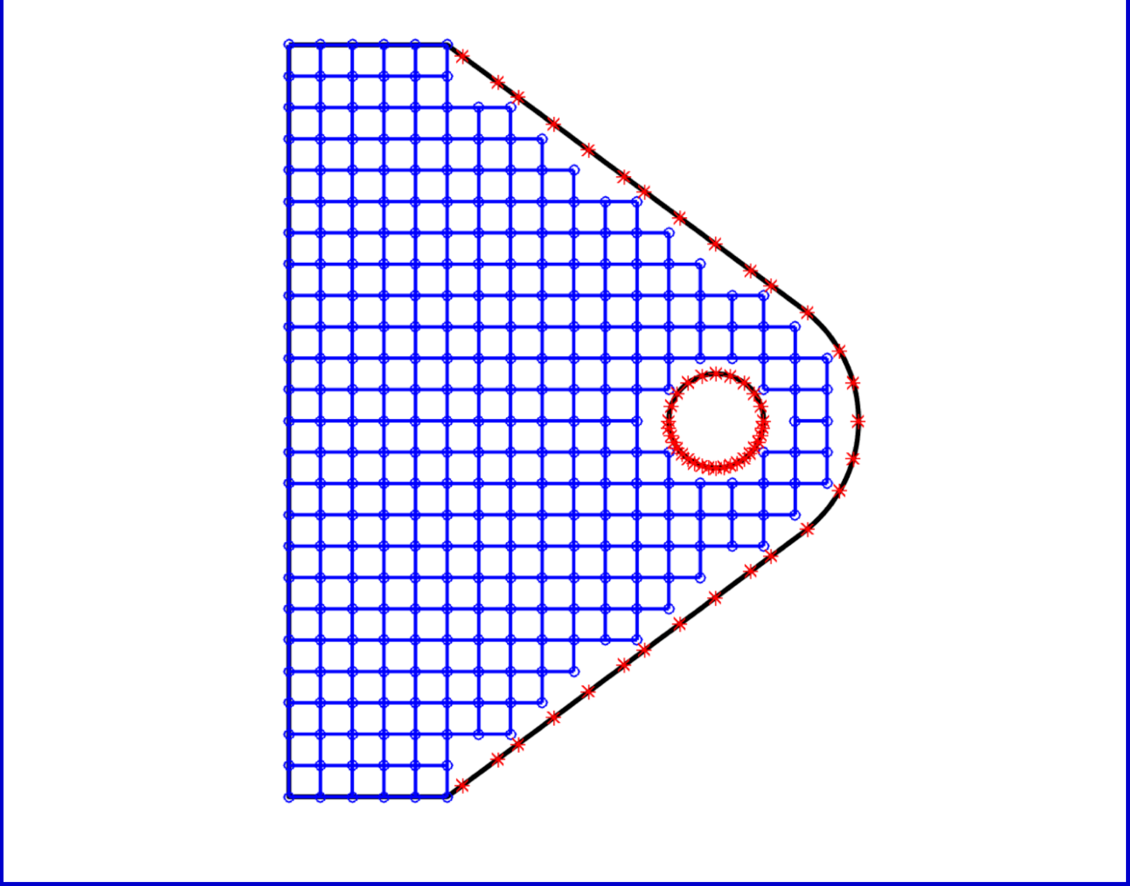
- **-- static analyses;**
- **-- dynamics:**
  calculations of frequencies and mode shapes,
  mode superposition &
  direct time integration

A simple illustrative example analysis of a bracket

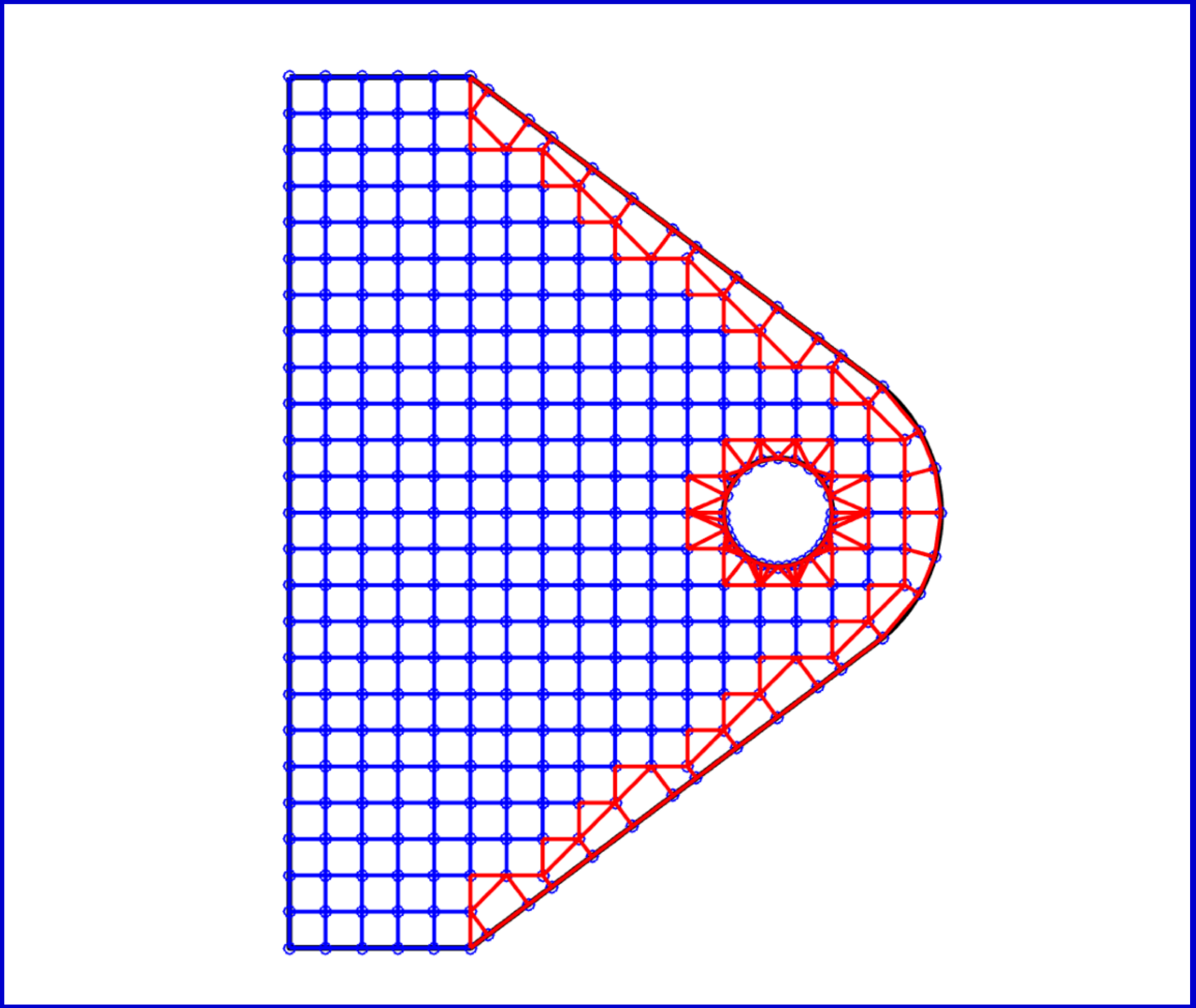




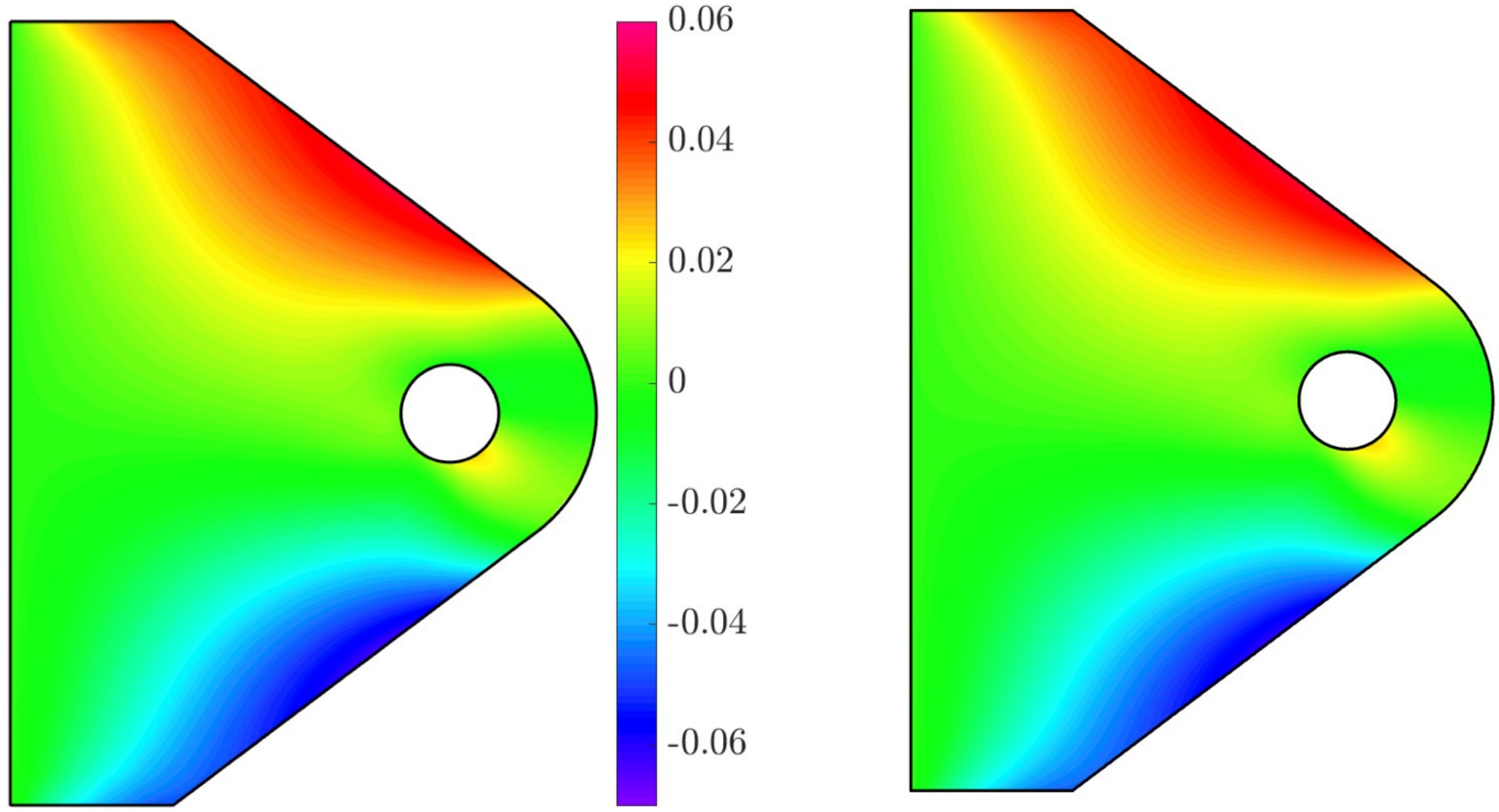
AMORE Step 1: The Cartesian mesh of cells



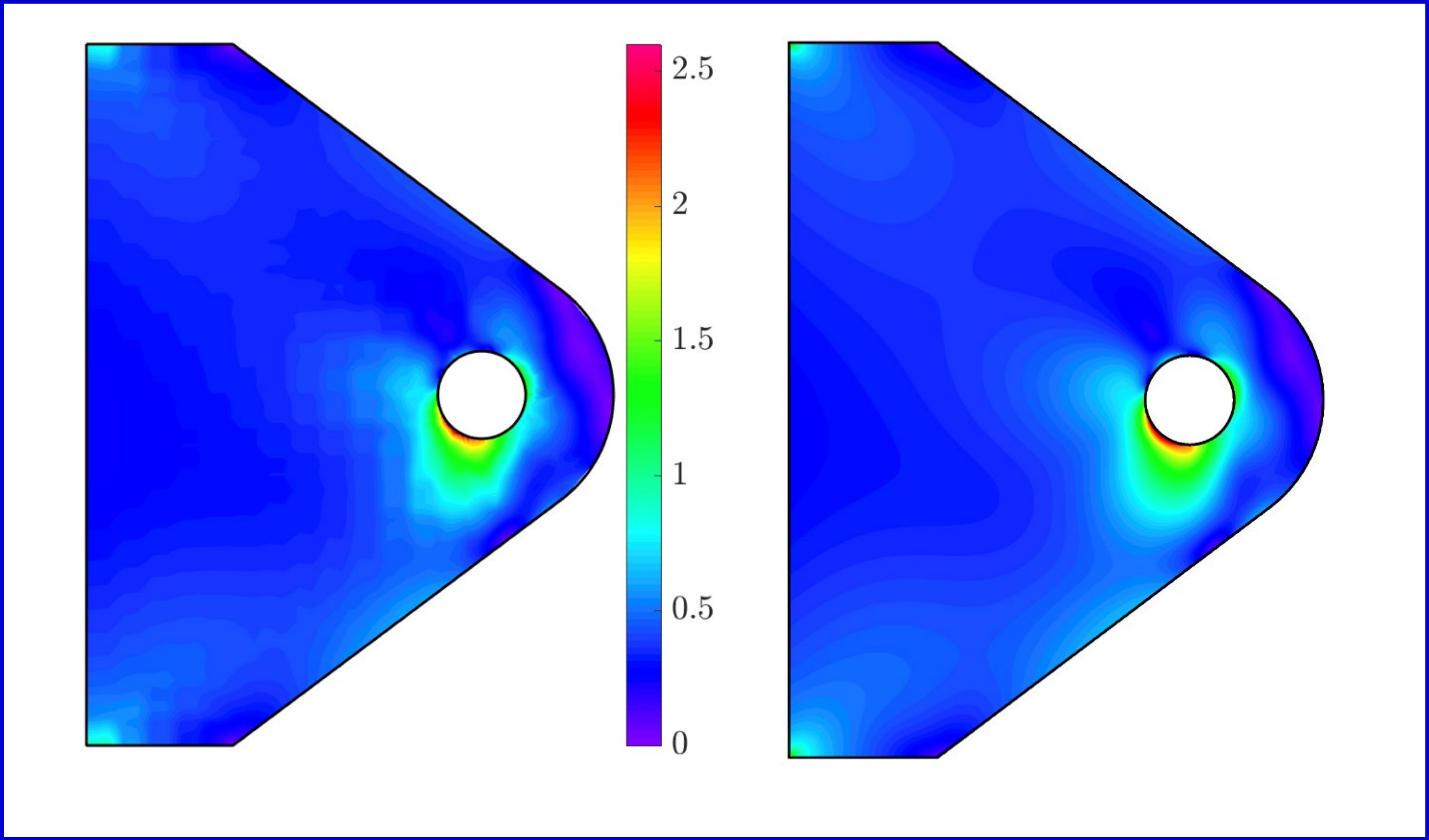
Step 2: Boundary is discretized; square cells outside and cutting the boundary are removed



Step 3: Square cells are turned into 4-node finite elements, empty spaces are filled with overlapping elements



Predicted hor. displacement left AMORE, right ref. solution



Predicted effective stress
left AMORE; right ref. solution

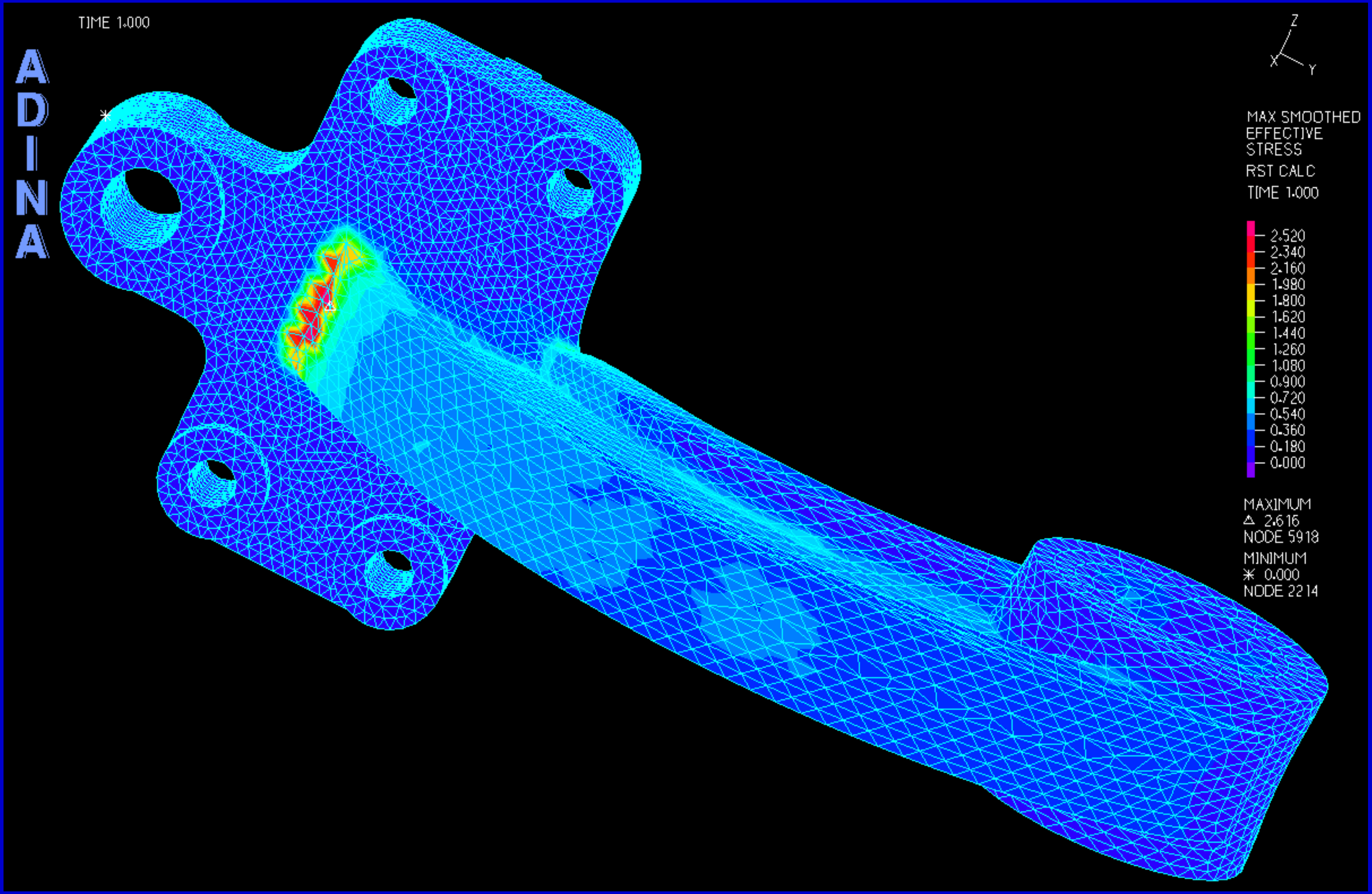
Overlapping elements

- Topics to be discussed --
- Different elements exist for 2D and 3D analyses
- All possess inherently the same properties of “distortion- insensitivity”
- Of most interest are 3D analyses, but 2D solutions also show element properties

Solutions

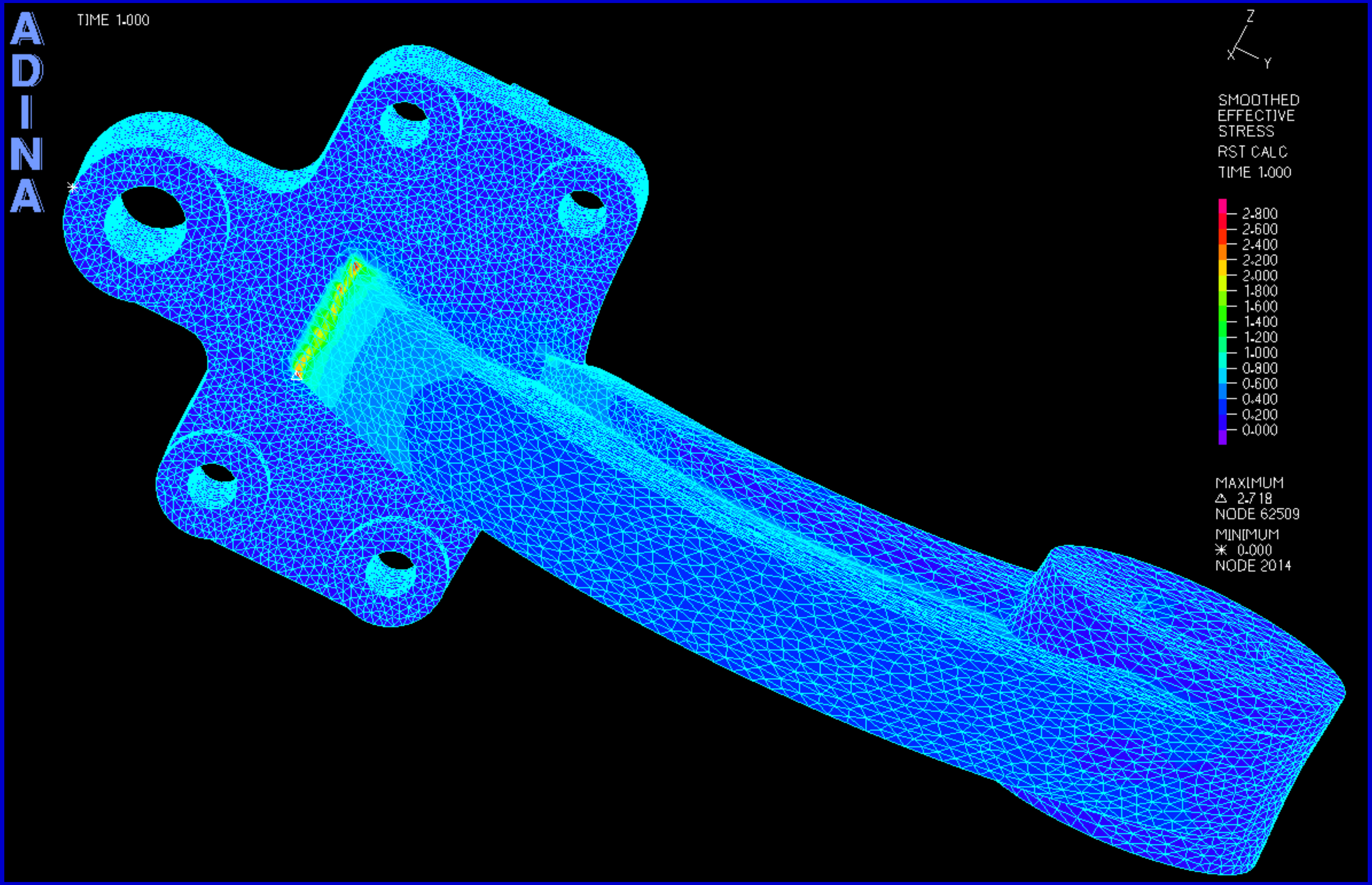
For solutions, special meshing procedures are needed, in particular for 3D solutions – some such schemes are available in ADINA, a software product of Bentley Systems, Inc.

Next we show an example analysis of a 3D bracket, AMORE versus traditional analyses



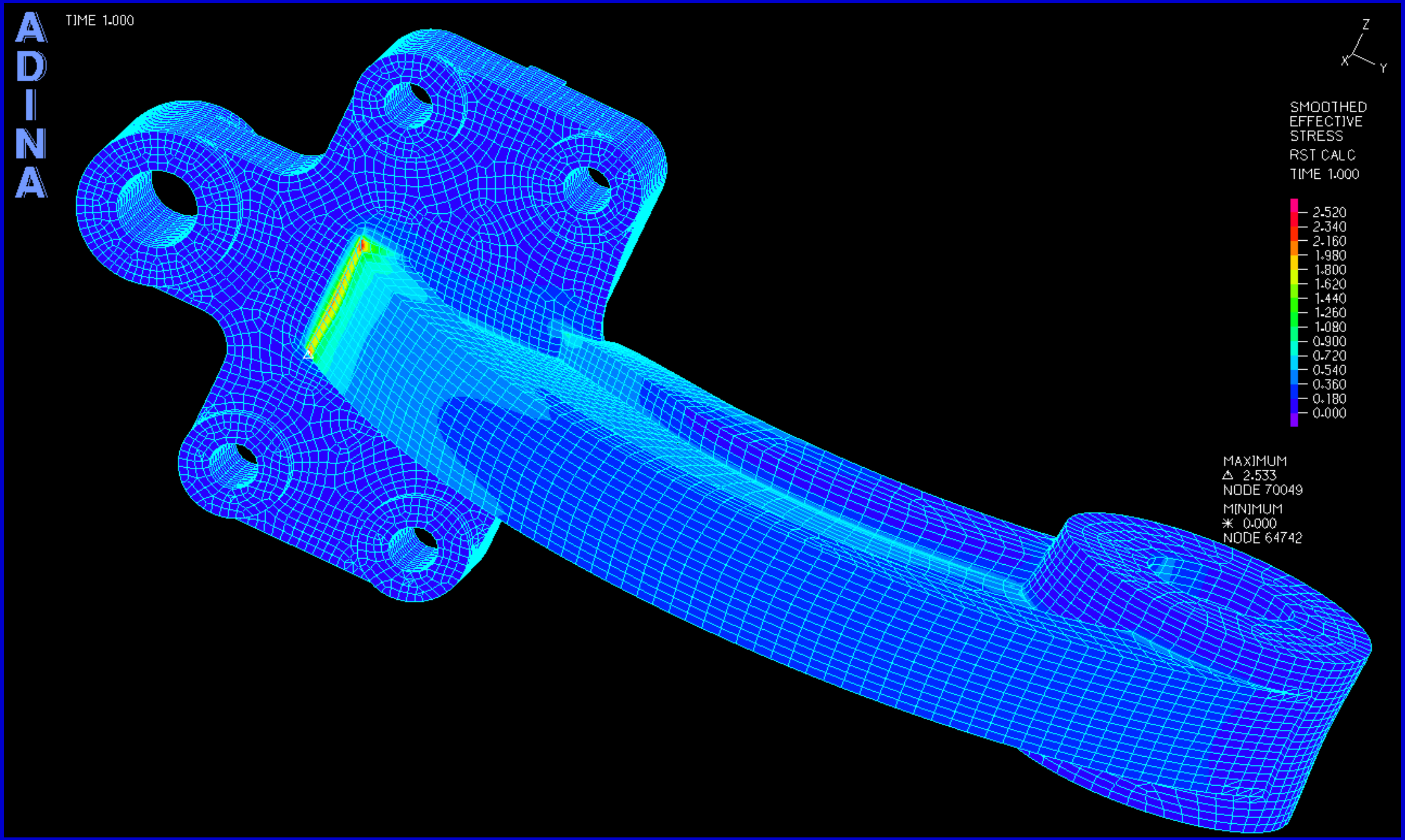
AMORE analysis using 8-node brick elements with incompatible modes and 4-node overlapping tetrahedral elements; number of eqns =122,730; max. y-displ = -22.25; max. von Mises stress = 2.62; solution time 7 sec.

Courtesy Bentley Systems



A traditional analysis using 10-node tetrahedral elements; Number of eqns = 739, 635; max. y-displ. = -22.42; max. von Mises stress = 2.72; solution time 60 sec; AMORE 8 times faster

Courtesy Bentley Systems



A traditional analysis using 27-node hexahedral elements. Number of eqns = 1,197, 816; max. y-displ. = – 22.48; max. von Mises stress = 2.53; solution time 242 sec; AMORE 35 times faster

Courtesy Bentley Systems

The OFE formulation

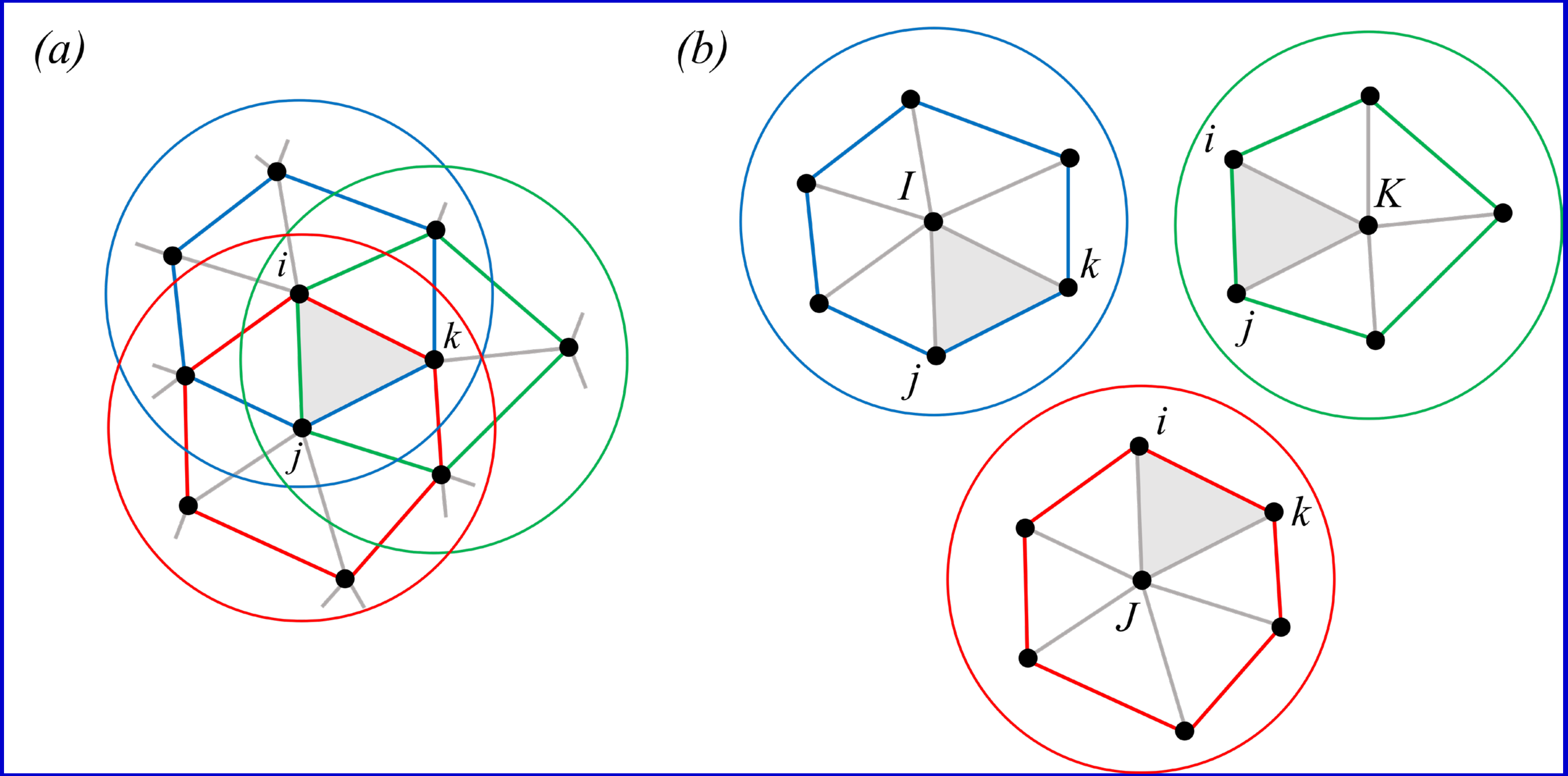
The OFE formulation is a further development of the ‘Method of Finite Spheres (MFS)’, and is also related to the works of --

I. Babuška, Q. Zhang, U. Banerjee, J.T. Oden, et. al.
on the ‘generalized finite element method’

but --

the OFE are formulated with ‘general equations’ (not specific to a specific element) giving continuous strain fields within each element, are stable, give reasonable condition numbers, and are used directly like traditional finite elements --- and in AMORE

The concept of the OFE formulation



The 3 polygonal elements I, J, K are overlapping in the shaded triangular region i-j-k; each polygonal element has as its "parent" a sphere of the Method of Finite Spheres

For the formulation of the OFE --

A polygonal element is placed at each node *i, j, k* – corresponding to each of the three spheres (in 2D, disks)

At each node, a nodal function with unknown parameters is used as in the MFS (polynomials, or other ...)

The displacements of the polygonal element on the element *i-j-k* are interpolated as usual but "weighted"

All interpolations are polynomials and usual FE procedures can be used, for the assemblage of elements, etc.

General Interpolation

$$u(\mathbf{x}) = \sum_I h_I\, \psi_I$$

$$\psi_I = \sum_K \varphi_K^I u_K$$

Hence

$$u(\mathbf{x}) = h_I \varphi_K^I u_K \;\; \boldsymbol{or} \;\; u(\mathbf{x}) = \rho_K u_K$$

with

$$\rho_K = h_I \varphi_K^I$$

$$u_K = a_{K1} + a_{K2} x_K + a_{K3} y_K + a_{K4} x_K y_K + \cdots$$

The polygonal element field

$$\psi_I = \sum_K \phi_K^I u_K$$

We want ϕ_K^I to be

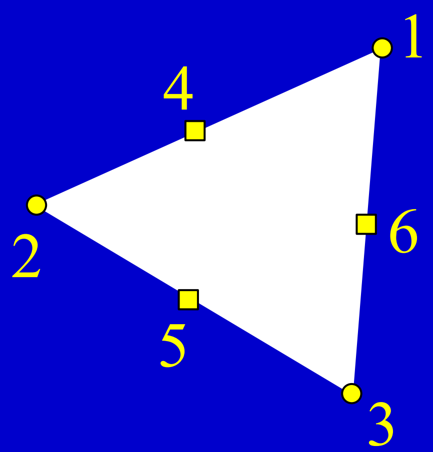
-- continuous over the polygonal element

-- easily integrable; hence we use polynomials (the MFS uses rational functions)

-- and $\sum_K \phi_K^I = 1$ for any *I*

We utilize the quadratic finite element functions to construct ϕ_K^I

$$\phi_K^I = \sum_{i \in QUADn} \hat{h}_i \, \hat{\phi}_{Ki}^I$$



$$\phi_1^1 = \hat{h}_1 + \hat{h}_4\left(0.5 - \beta\right) + \hat{h}_6\left(0.5 - \beta\right)$$

$$\phi_2^1 = \hat{h}_2 + \hat{h}_4\left(0.5 + \beta\right) + \hat{h}_5\left(0.5\right)$$

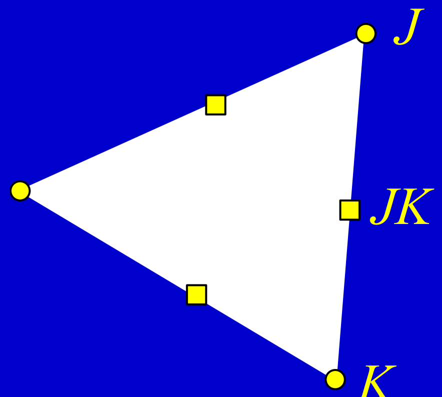
$$\phi_3^1 = \hat{h}_3 + \hat{h}_5\left(0.5\right) + \hat{h}_6\left(0.5 + \beta\right)$$

$$\phi_1^1 + \phi_2^1 + \phi_3^1 = 1$$

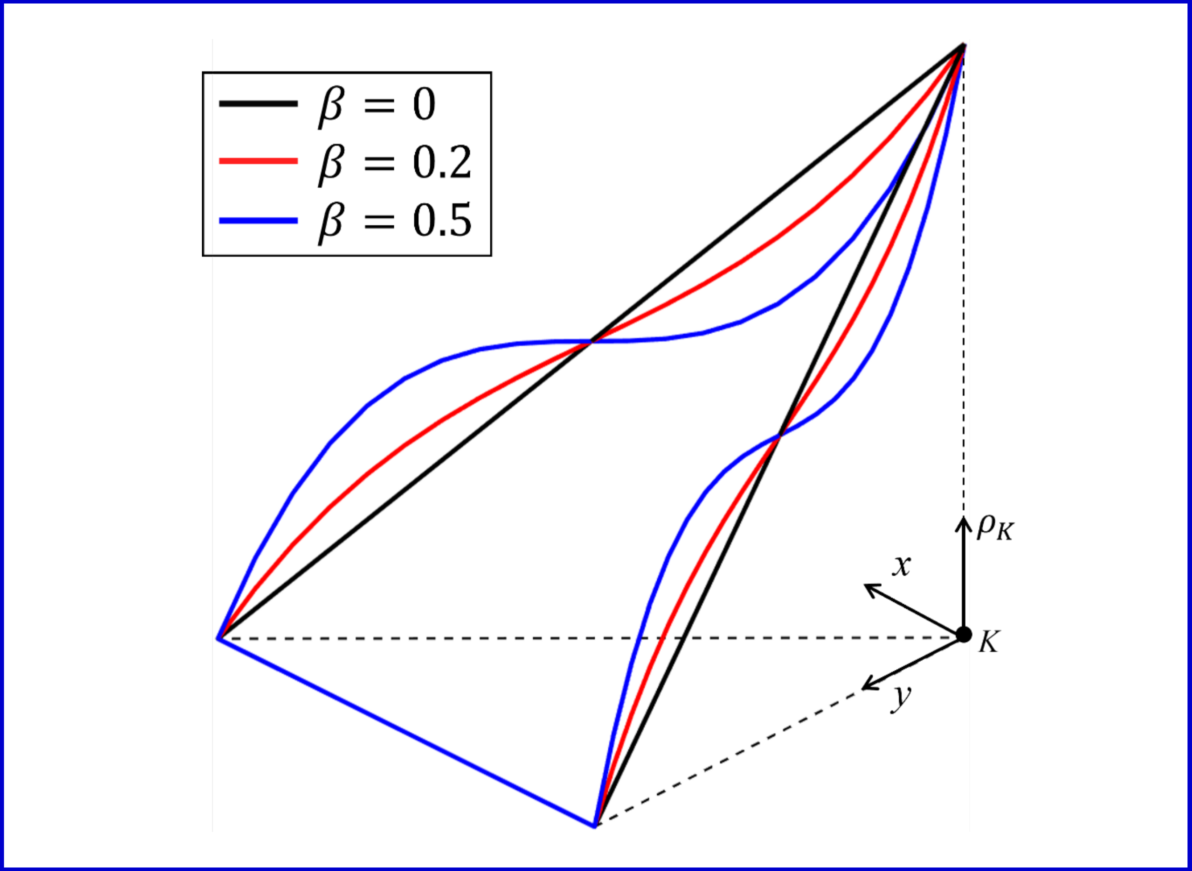
The 2D interpolation

$$u(\mathbf{x}) = \rho_K \, u_K$$

$$\rho_K = h_K + \beta \sum_{I \neq K} \left(h_I - h_K \right) \hat{h}_{IK}$$

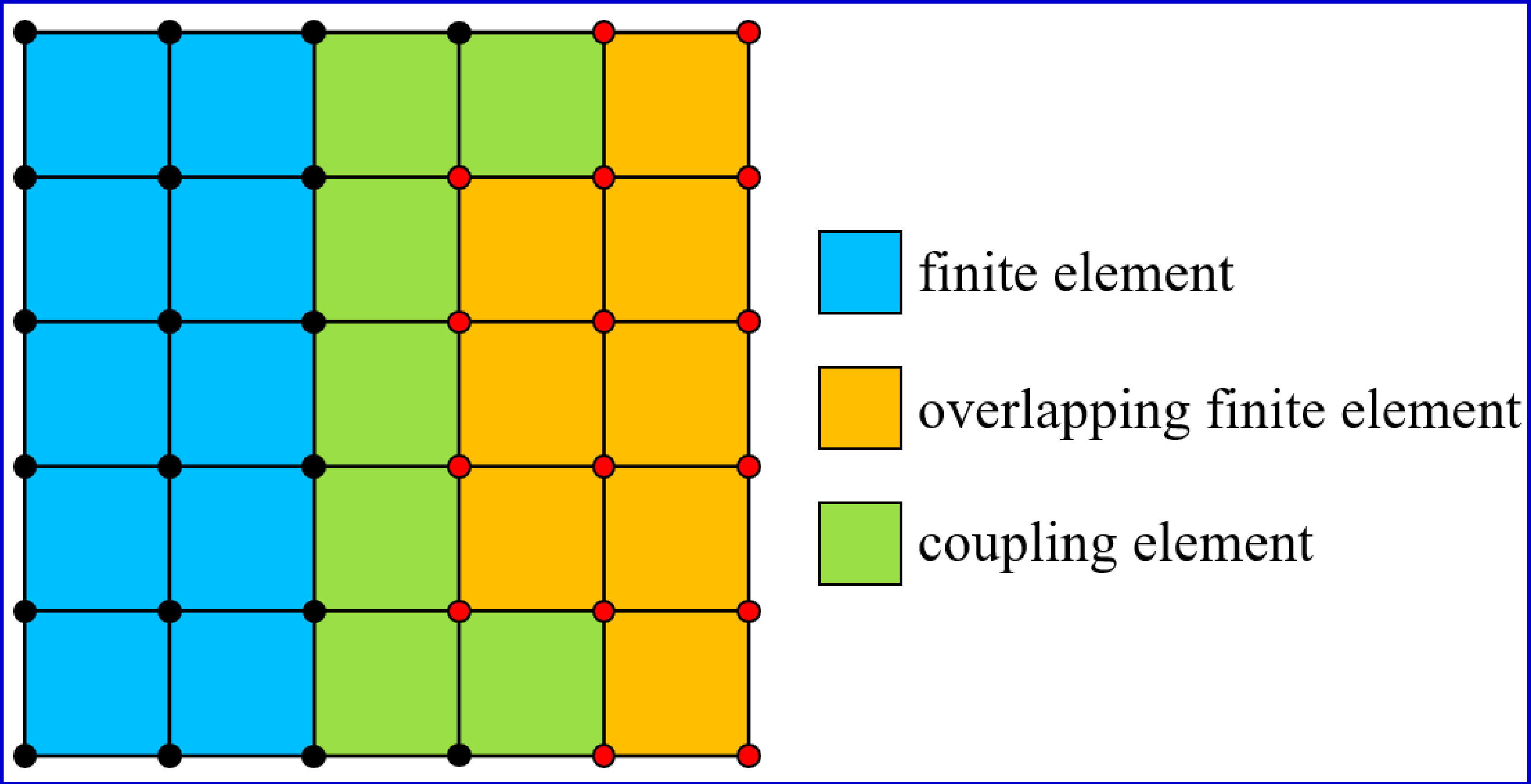


The "Trad. FEM" and the "FEM with Interpolation Covers" are special cases of the OFEM

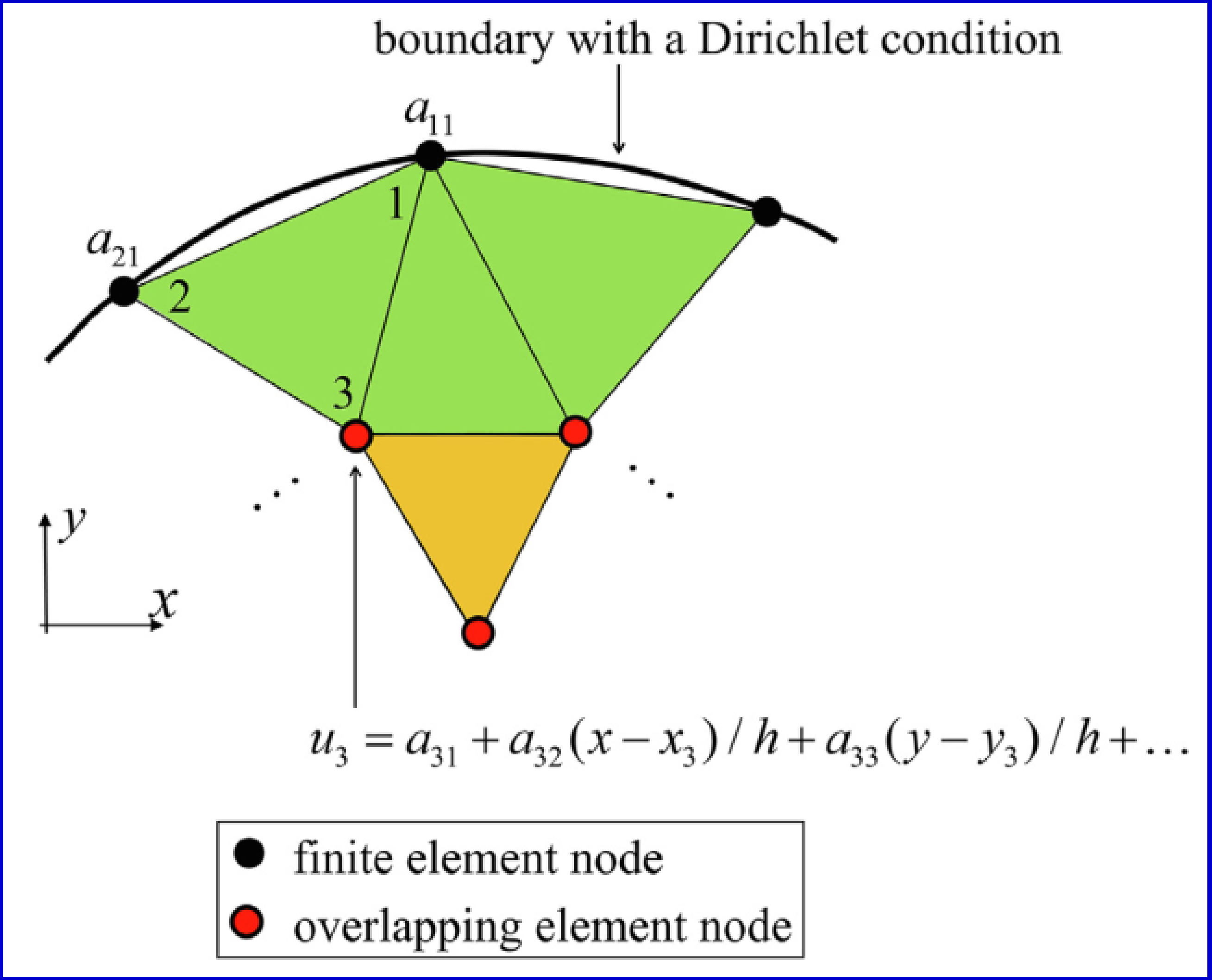


$$\text{with } \beta = 0, \quad u(\mathbf{x}) = \sum_K \rho_K u_K = \sum_K h_K u_K$$

Coupling elements

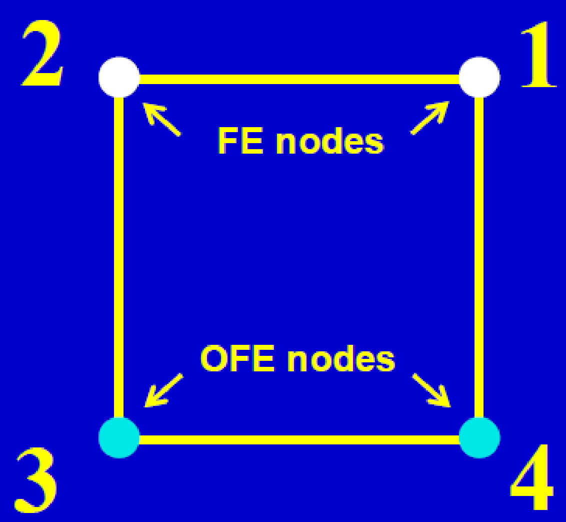


Coupling elements are used for the transitions from regular finite elements to overlapping elements



Coupling elements are also used to impose the displacement boundary conditions as in the traditional finite element method

Example: 4 – node coupling element formulation



u_i : usual dof

$\tilde{\underline{u}}_i$: additional dofs

FEn = {1,2}

OFEn = {3,4}

node 1 : u_1

node 2 : u_2

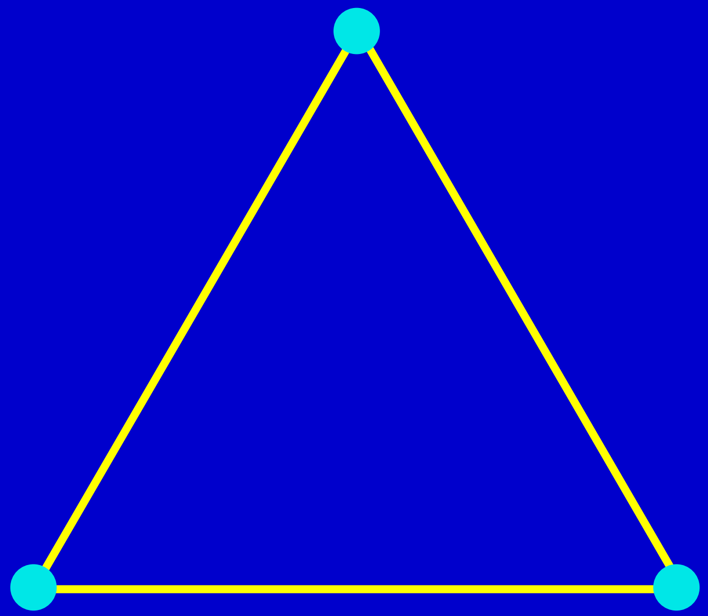
node 3 : $u_3, \tilde{\underline{u}}_3$

node 4 : $u_4, \tilde{\underline{u}}_4$

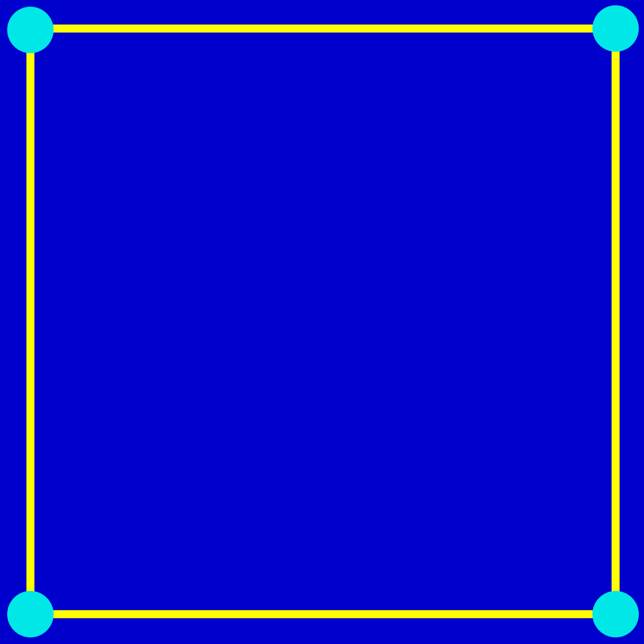
For implementation – example: The 4-node quad coupling element matrices, linear basis

$$u = \mathbf{H}\mathbf{u} = \begin{bmatrix} \rho_1^{FEn} & \rho_2^{FEn} & \rho_3^{OFEn} & \rho_4^{OFEn} & \dots \\ \dots & \tilde{\rho}_3^{OFEn}\,x_3 & \tilde{\rho}_3^{OFEn}\,y_3 & \tilde{\rho}_4^{OFEn}\,x_4 & \tilde{\rho}_4^{OFEn}\,y_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \tilde{u}_3^1 \\ \tilde{u}_3^2 \\ \tilde{u}_4^1 \\ \tilde{u}_4^2 \end{bmatrix}$$

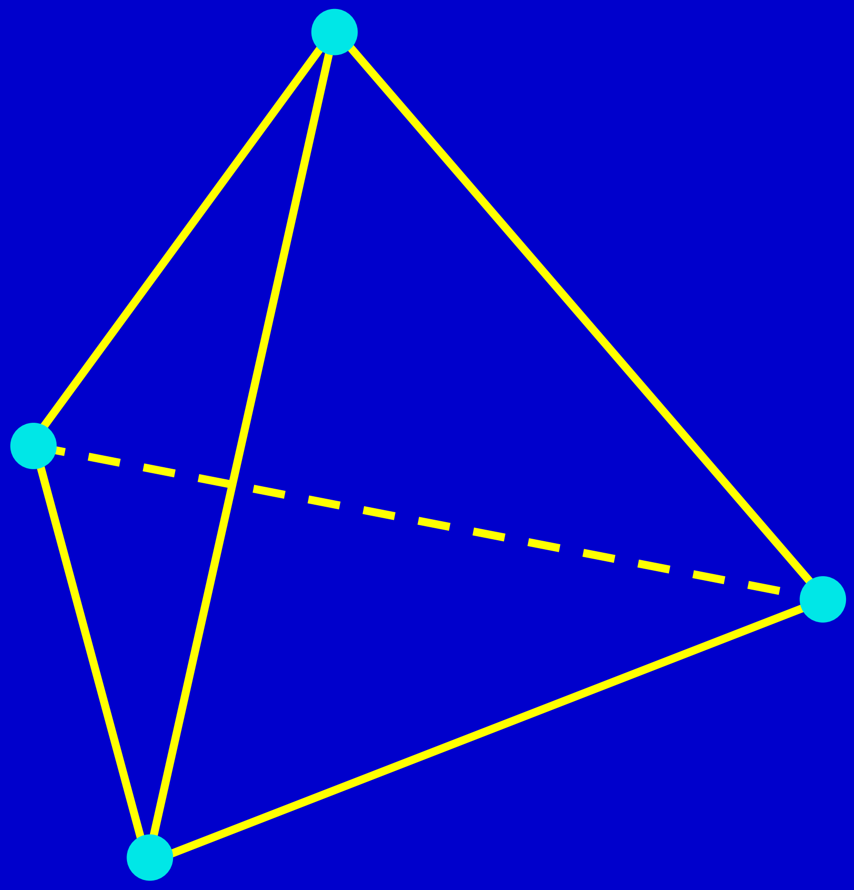
The formulation is general, so we have



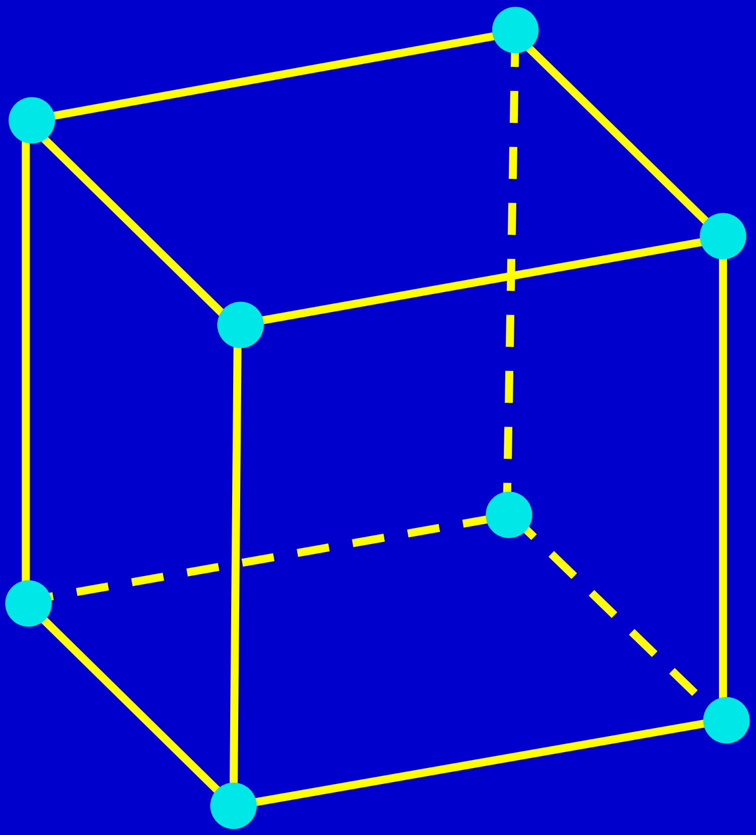
3-node triangular overlapping element



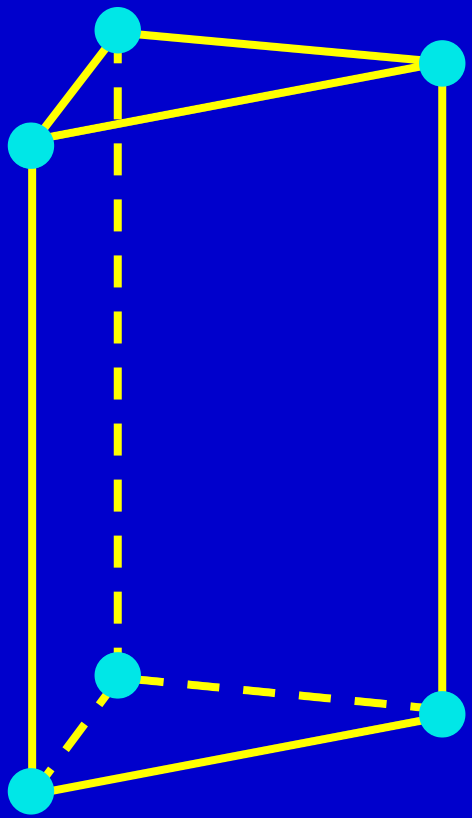
4-node quadrilateral overlapping element



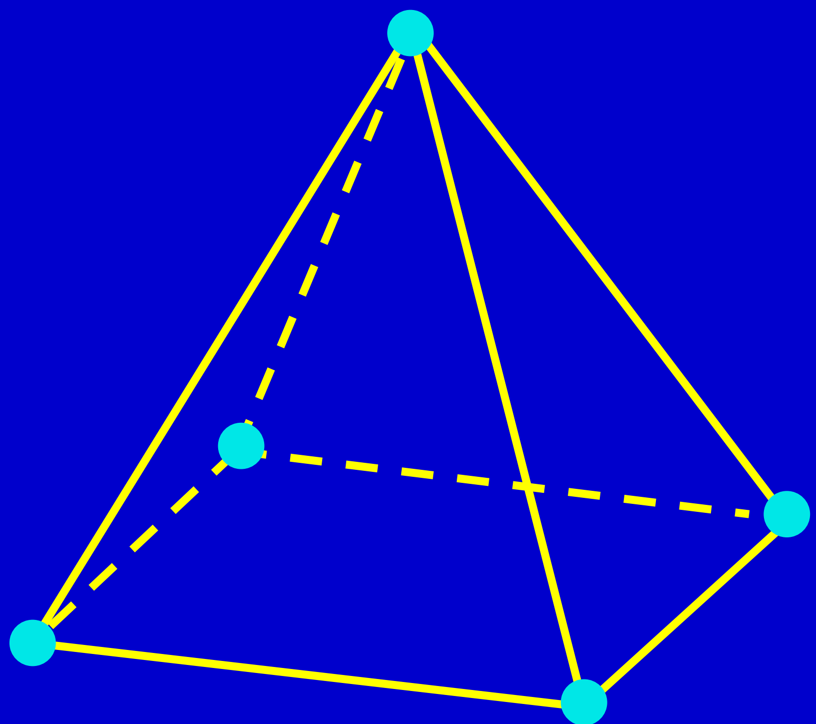
4-node tetrahedral overlapping element



8-node brick overlapping element



6-node prism overlapping element



5-node pyramid overlapping element

Stability of OFEM

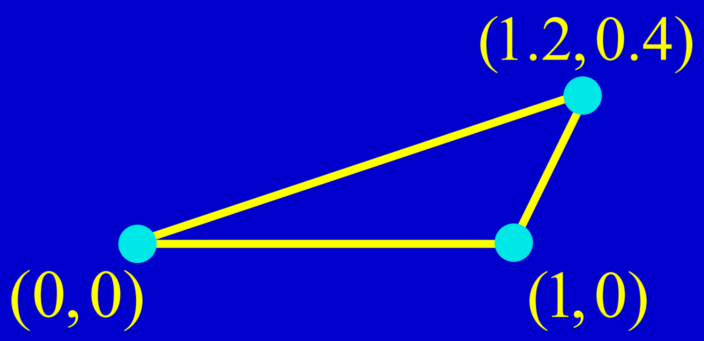
We need --

no spurious mode in any element – then the stiffness matrix is positive definite

And we have --

The overlapping elements contain no spurious zero energy mode

Zero energy mode test, distorted triangle



Young's modulus = 2 × 10^9 Pa
Poisson's ratio = 0.3
Plane stress condition
No displacement BC imposed
β = 0.03

Basis	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Linear	-2.54E-07	3.11E-08	4.02E-07	7.58E+04	1.43E+05	2.90E+05	5.24E+05	9.76E+05
Bilinear	-2.09E-07	-4.66E-08	1.94E-07	5.49E+01	8.55E+01	4.86E+03	1.33E+04	1.19E+05
Quadratic	-3.09E-07	4.23E-09	1.49E-07	6.21E-01	2.61E+00	5.00E+00	5.61E+00	3.94E+01

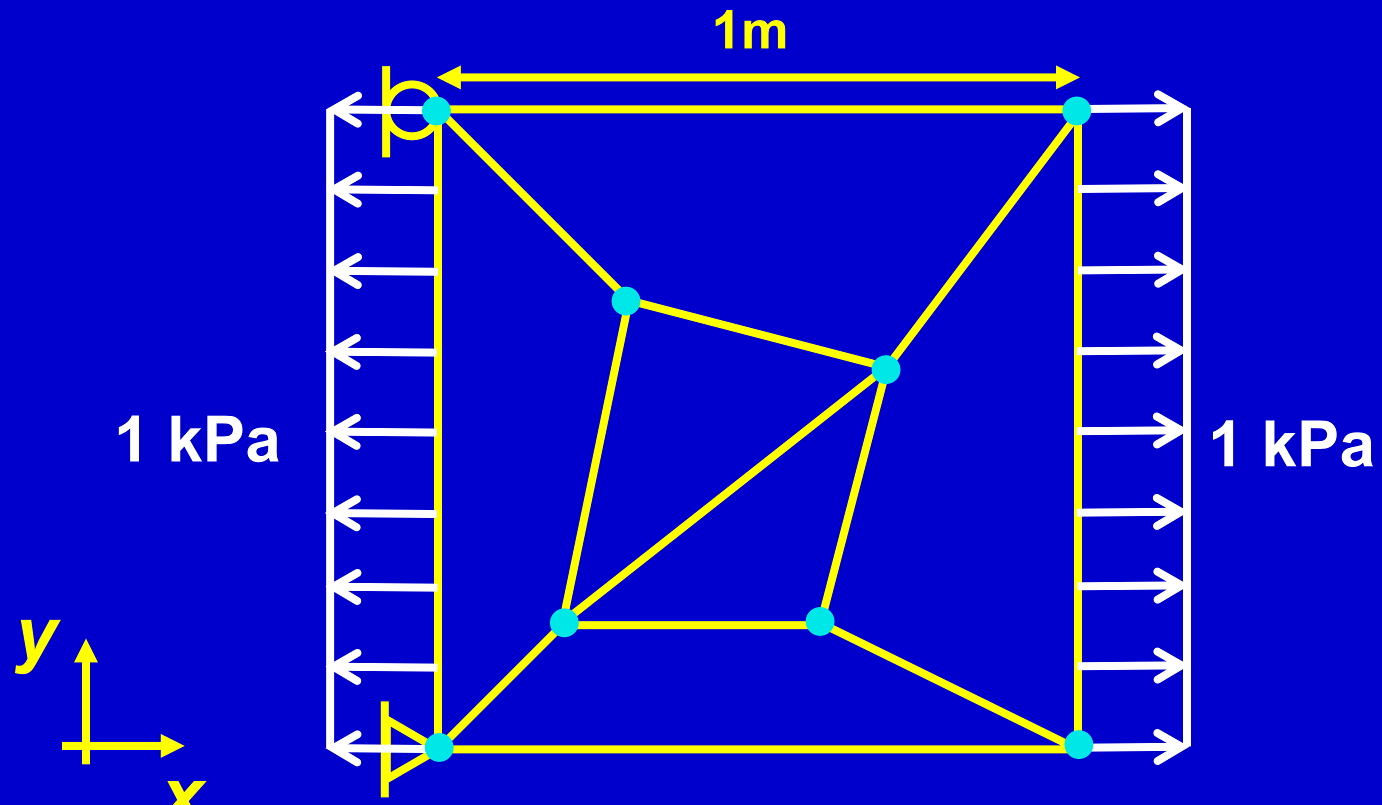
Completeness of OFE

An element needs to be complete for convergence of solutions

The overlapping elements are complete:

Each element contains the rigid body mode displacements and can represent the constant stress states; all OFE pass the patch test

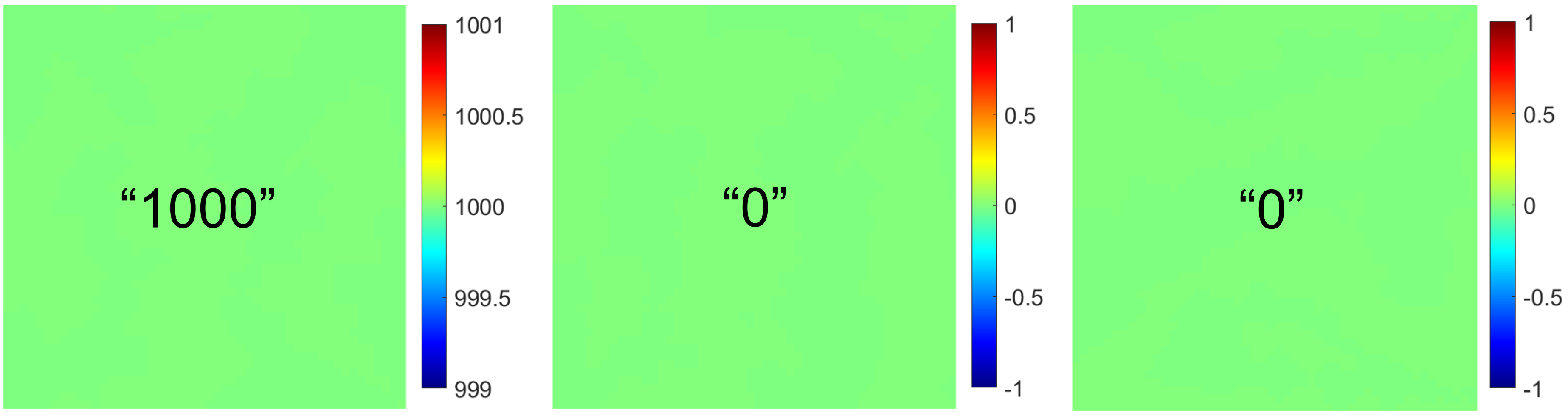
Patch test



Young's modulus = 200 GPa
Poisson's ratio = 0.3
Plane stress condition

The 2D overlapping elements are used

Stress solutions



left τ_{xx} center τ_{yy} right τ_{xy}

The constant stress state is reproduced

Geometric distortion insensitivity

The overlapping elements – other than the regular finite elements – preserve polynomial completeness when distorted, consider

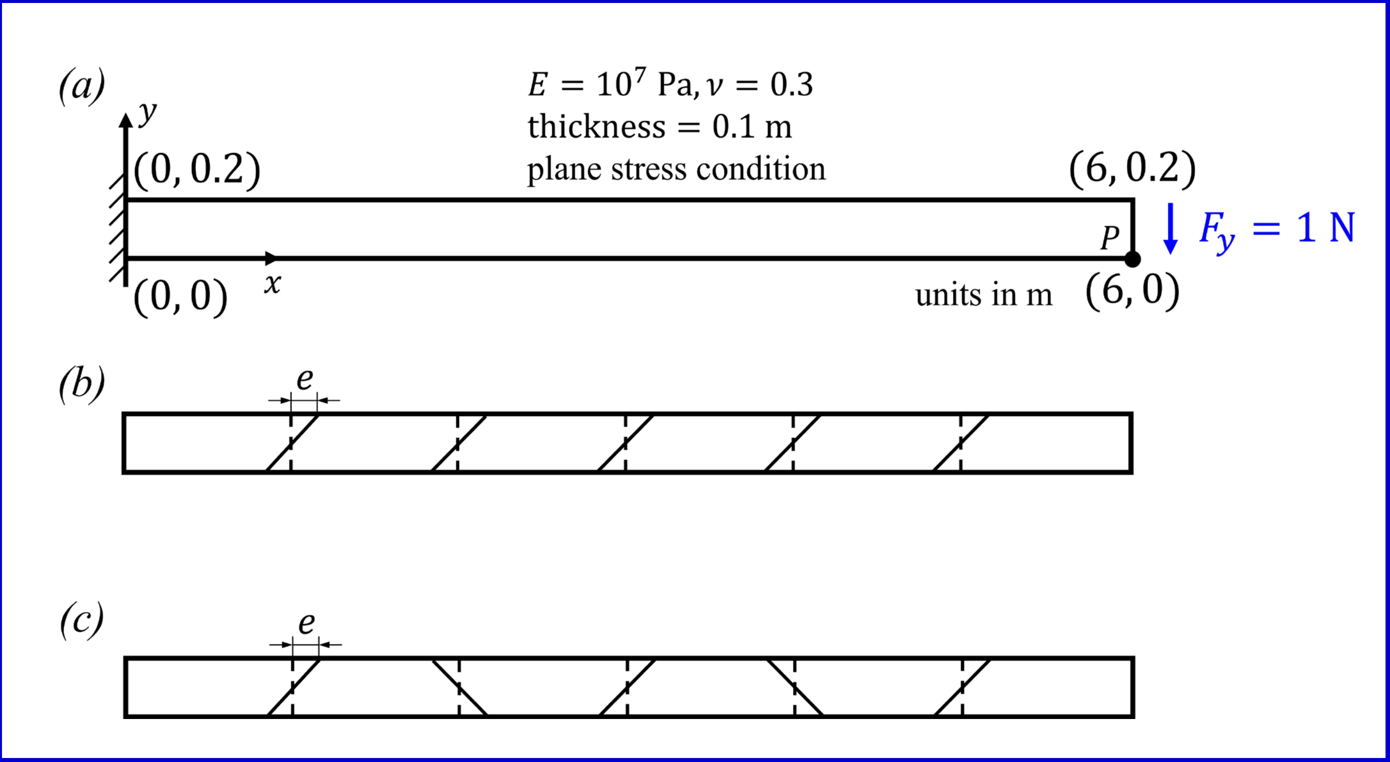
$$u_1 = u_2 = \cdots = u_K = p$$
$$u(\mathbf{x}) = \rho_K u_K = \rho_K p = p$$

with p a polynomial, see definition of u_K

Hence, the OFE are effective in filling the regions not covered by regular elements in the AMORE scheme

Example analysis

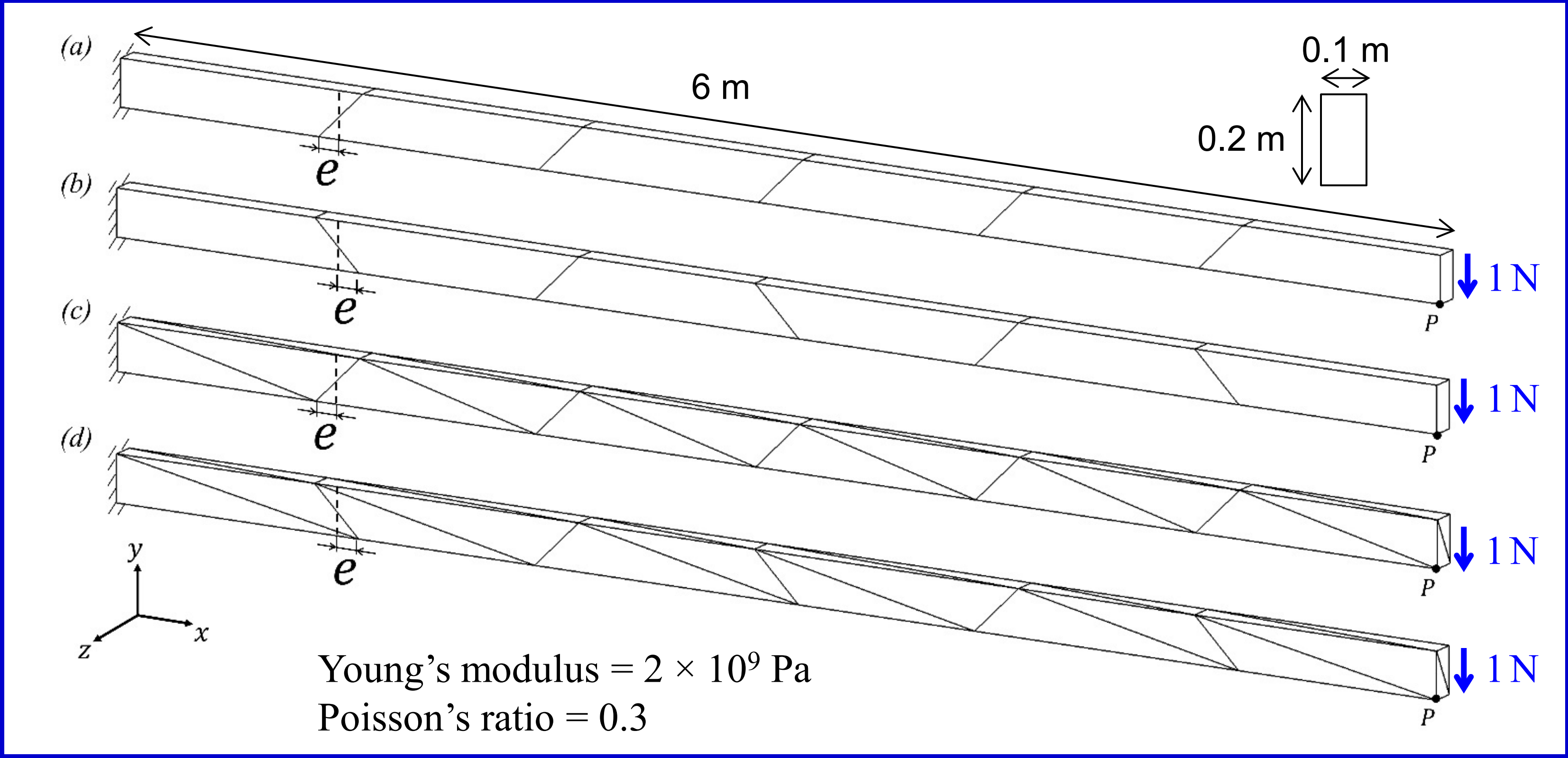
to illustrate the distortion insensitivity



9-node FE (72 dofs)	e = 0 (m)	e = 0.1	e = 0.2	e = 0.3	e = 0.4
parallelogram	0.9901	0.9813	0.9397	0.8770	0.8252
trapezoidal		0.9811	0.9234	0.8422	0.7966
Quadratic OFE (156 dofs)	**e = 0**	**e = 0.1**	**e = 0.2**	**e = 0.3**	**e = 0.4**
parallelogram	0.9909	0.9917	0.9925	0.9920	0.9905
trapezoidal		0.9913	0.9910	0.9903	0.9906

The 4-node OFE is distortion-insensitive

3D analyses of a slender beam



Traditional Finite elements

27-node FE 6×1×1 mesh (324 dofs)	e = 0 (m)	e = 0.1	e = 0.2	e = 0.3	e = 0.4
parallelogram	0.9784	0.9690	0.9267	0.8659	0.8169
trapezoidal		0.9682	0.9077	0.8296	0.7883
27-node FE 12×2×2 mesh (1,800 dofs)	e = 0	e = 0.1	e = 0.2	e = 0.3	e = 0.4
parallelogram	0.9907	0.9895	0.9848	0.9745	0.9551
trapezoidal		0.9900	0.9884	0.9872	0.9861

Overlapping elements

4-node quadratic OFE (768 dofs)	e = 0 (m)	e = 0.1	e = 0.2	e = 0.3	e = 0.4
Parallelogram	0.9807	0.9810	0.9808	0.9800	0.9791
Trapezoidal		0.9797	0.9782	0.9759	0.9728
8-node quadratic OFE (768 dofs)	e = 0	e = 0.1	e = 0.2	e = 0.3	e = 0.4
parallelogram	0.9876	0.9877	0.9879	0.9878	0.9876
trapezoidal		0.9876	0.9875	0.9873	0.9872

Two additional issues --

1/ Conditioning of governing equations when using OFE

The stiffness matrix is positive definite provided we use the coupling element to impose the displacement boundary conditions and $\beta > 0$

In fact, for the 2-node 1D, 3-node 2D and the 4-node 3D elements β can be zero (in this case the OFE corresponds to the finite element with covers)

2/ Computational costs

Numerical integrations --

Higher order numerical integration is required, but it is a small portion of the entire computational cost

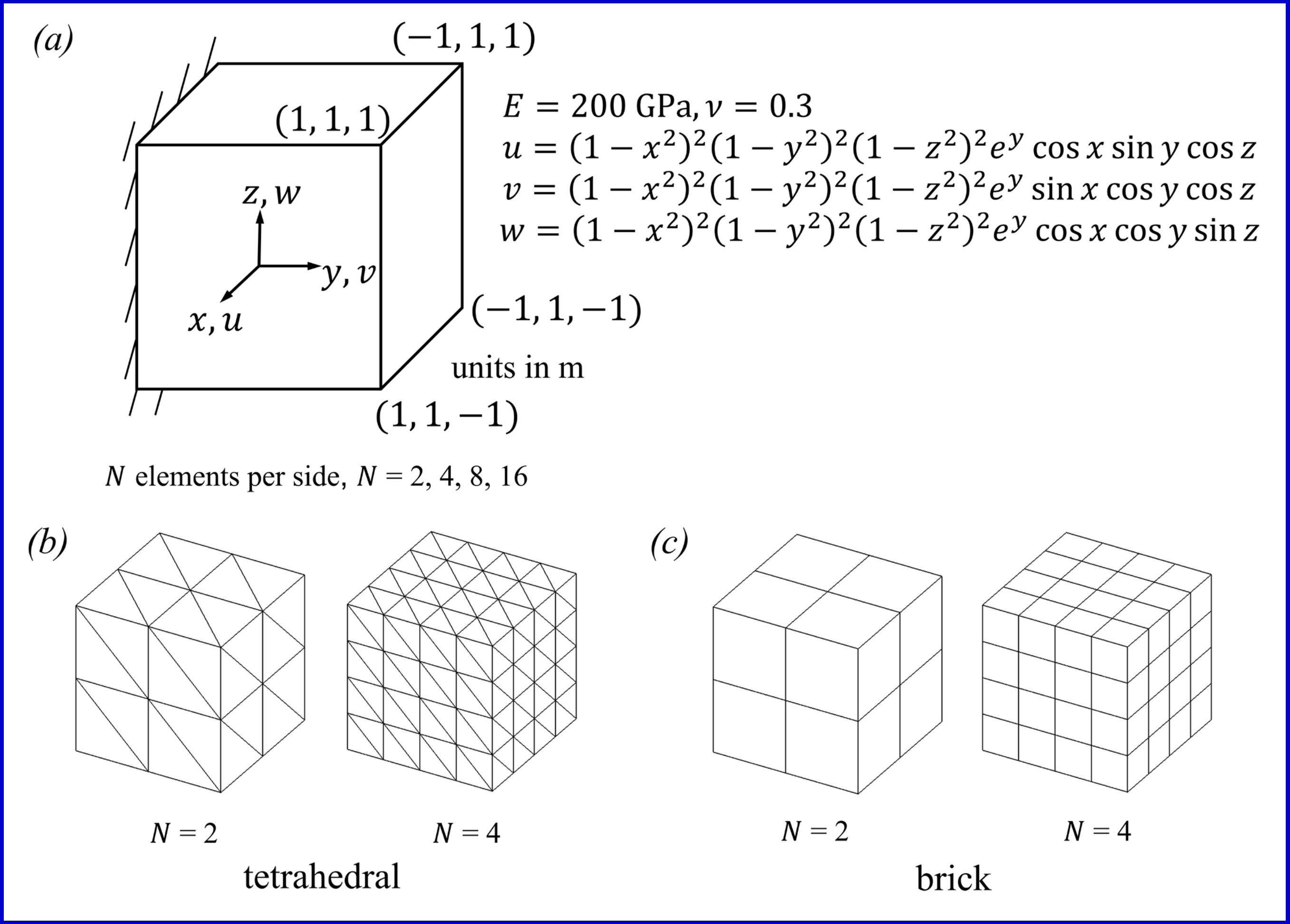
Solution of the algebraic equations --

The bandwidths due to OFE are larger than when using the traditional FEM, but a coarser mesh can frequently be used

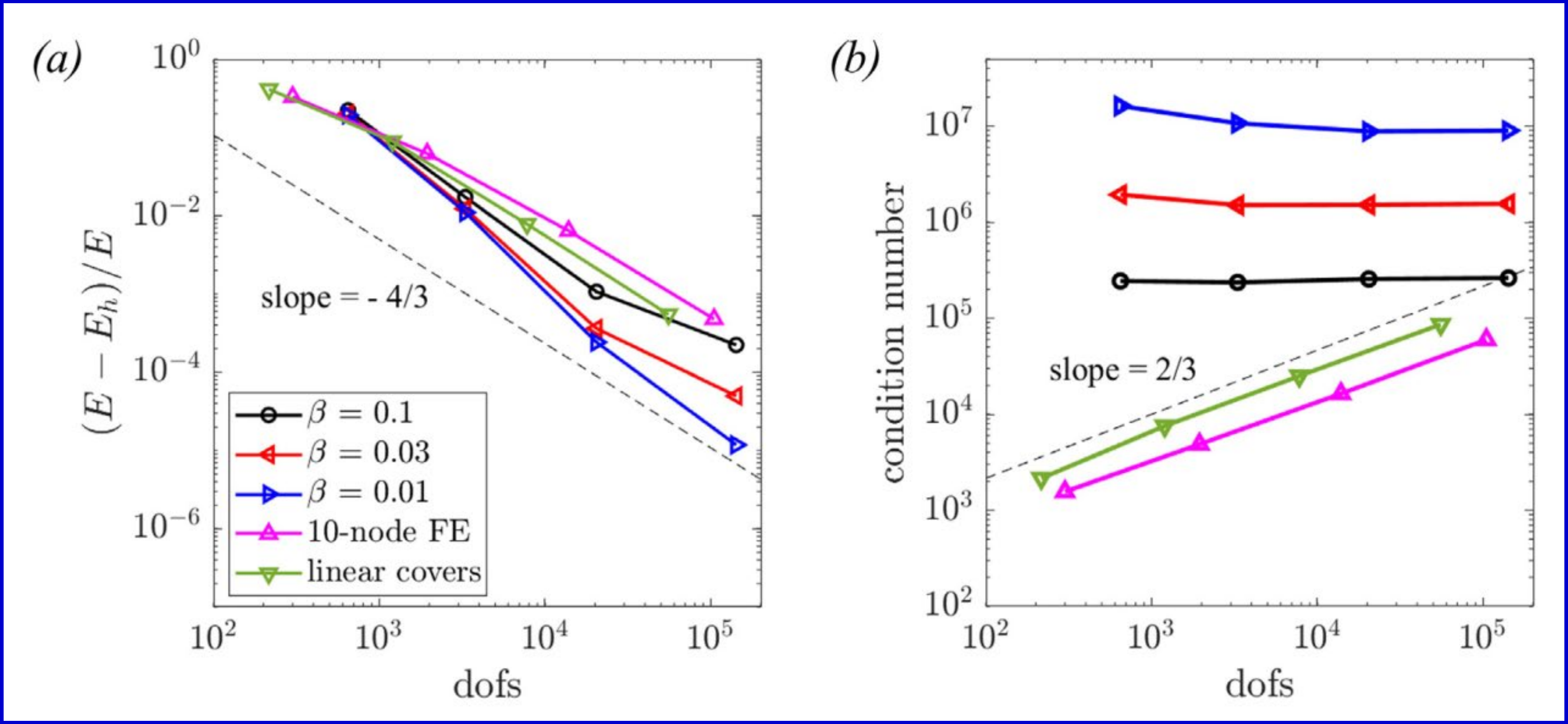
&

we have AMORE

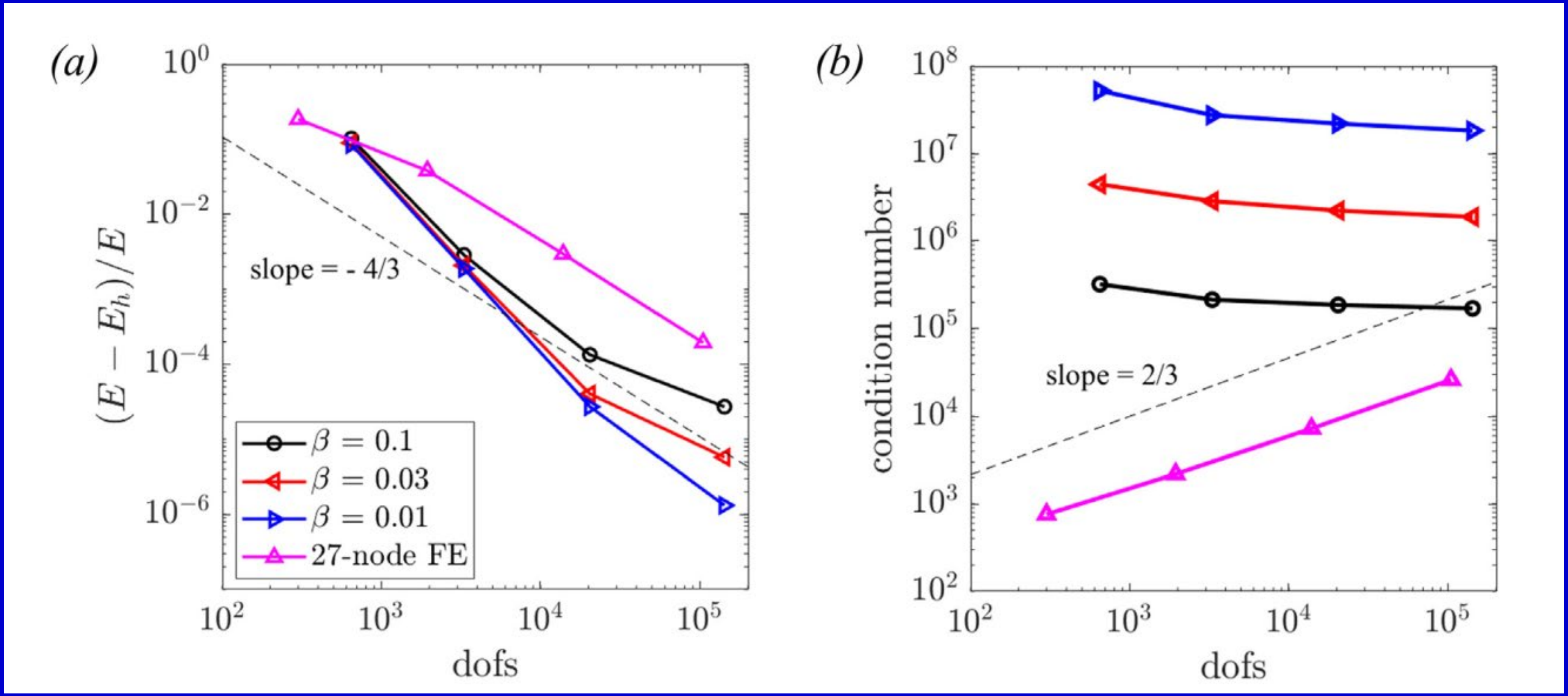
Convergence of solutions in 3D



Results using the 4-node tetrahedral overlapping element with the quadratic basis



Results using the 8-node brick overlapping element with the quadratic basis



Consider the solution of eigenproblems --

we are interested in solving:

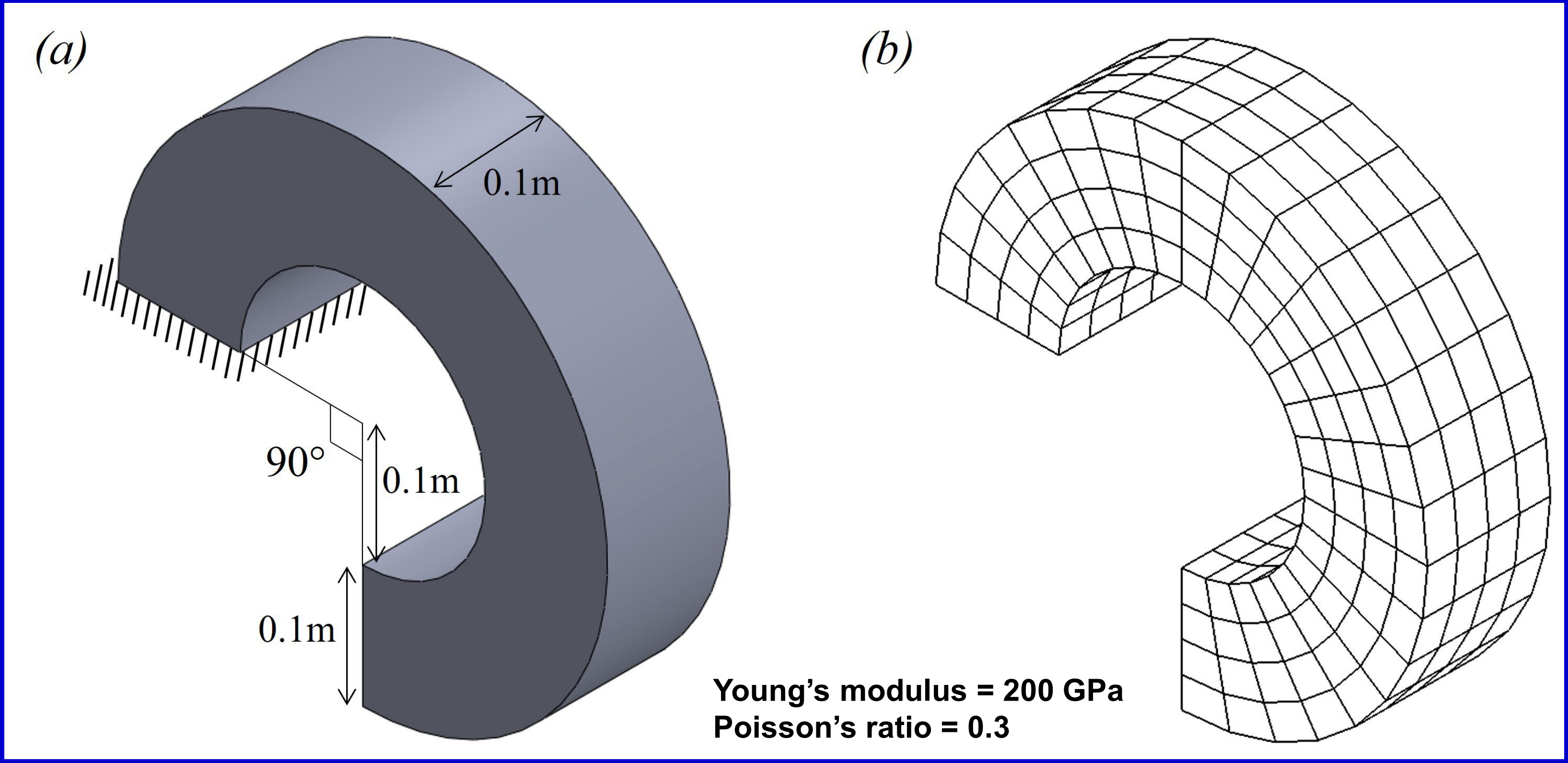
$$\mathbf{K}\boldsymbol{\varphi}_i = \lambda_i \mathbf{M}\boldsymbol{\varphi}_i$$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$$

with p = number of required eigenpairs

We use the enriched
Bathe Subspace Iteration method

Frequency analysis, an example solution

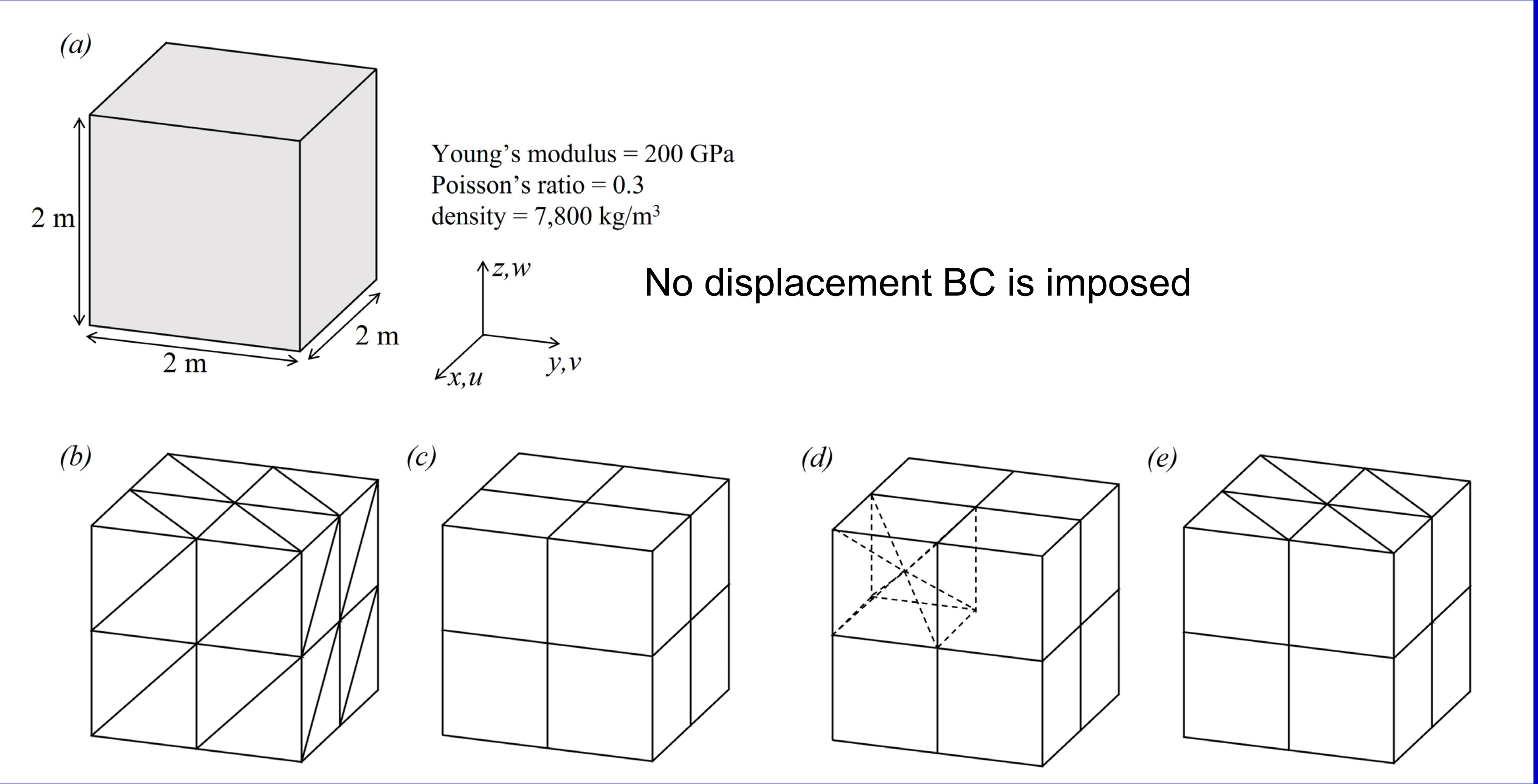


Calculation of lowest 20 frequencies; solution times normalized to Total CPU time used for the regular element solution

Element (m × m × 6m)	Log (rel. error of 1st freq.)	Numerical integration time	Total CPU time	Half-bandwidth of K	Total DOFs
8-node incompatible mode (24 × 24 × 144)	-3.06	9.0×10^{-3}	1.0×10^{0}	5,553	270,000
8-node linear OFE (10 × 10 × 60)	-3.07	3.1×10^{-3}	2.1×10^{-1}	3,795	87,483

Hence, we have an efficient solution using the overlapping element

Frequency solutions of a block in 3D



No displacement BC is imposed

Calculation of lowest 10 frequencies; solution times normalized to Total CPU time used for regular element solution

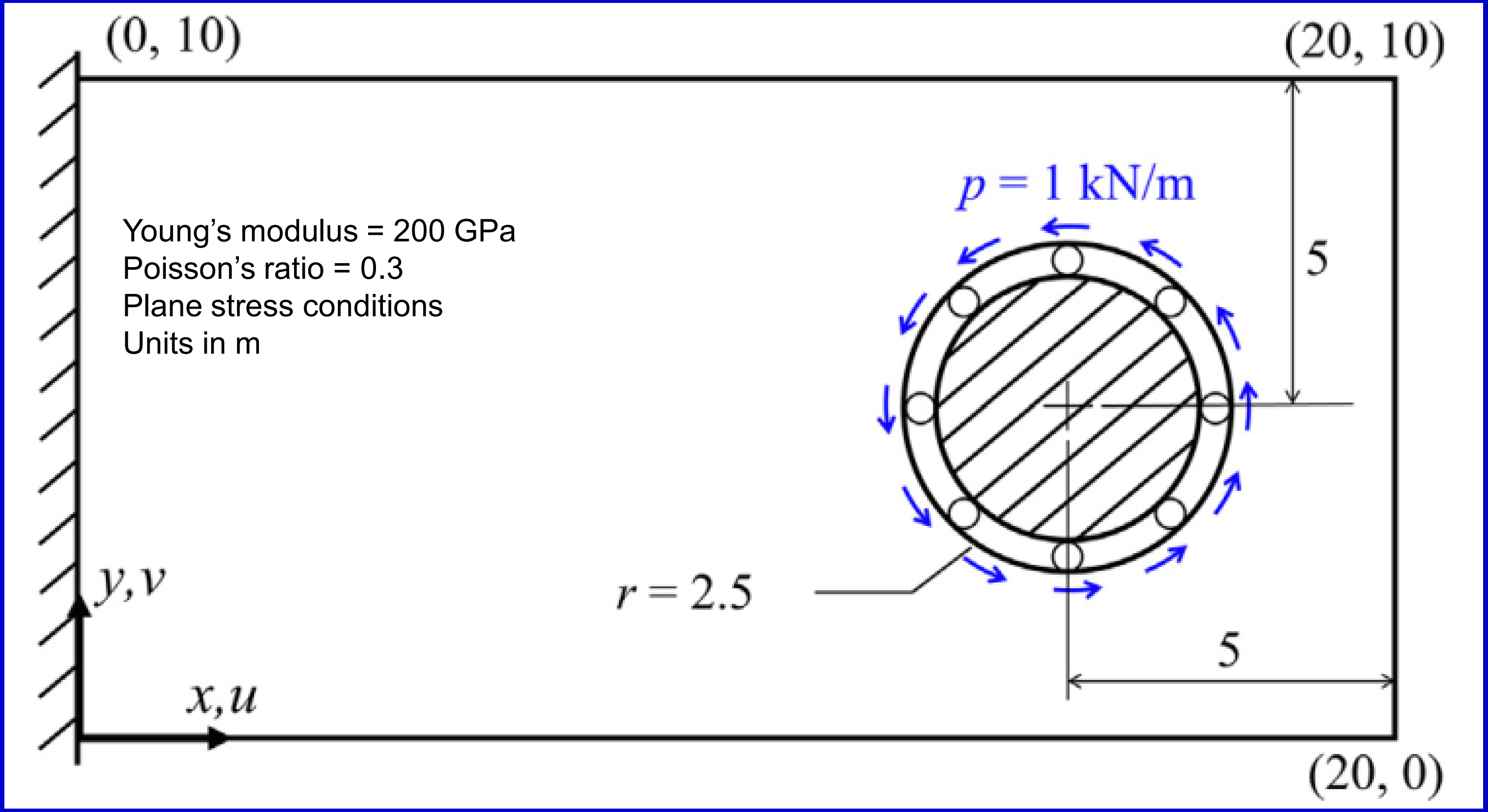
Element (N of mesh)	Log (rel. error of 1st freq.)	Numerical integration time	Total CPU time	Half-bandwidth of K	Total degrees of freedom
8-node ICM (32)	-3.02	1.9×10^{-3}	1.0×10^{0}	9,510	107,811
4-node linear OFE (8)	-3.06	1.4×10^{-4}	4.7×10^{-3}	792	8,748
5-node linear OFE (5)	-3.01	6.9×10^{-5}	2.3×10^{-3}	1,644	4,092
6-node linear OFE (6)	-3.13	6.1×10^{-5}	1.9×10^{-3}	852	4,116
8-node linear OFE (4)	-3.01	2.1×10^{-5}	3.7×10^{-4}	744	1,500

The OFEM is efficient

Some solutions using AMORE

- The objective is to show the wide applicability of the AMORE scheme in statics and dynamics
- The efficiency and accuracy of the solutions obtained is illustrated
- We showed already the analysis of a 3D bracket under static loading and add now further solutions

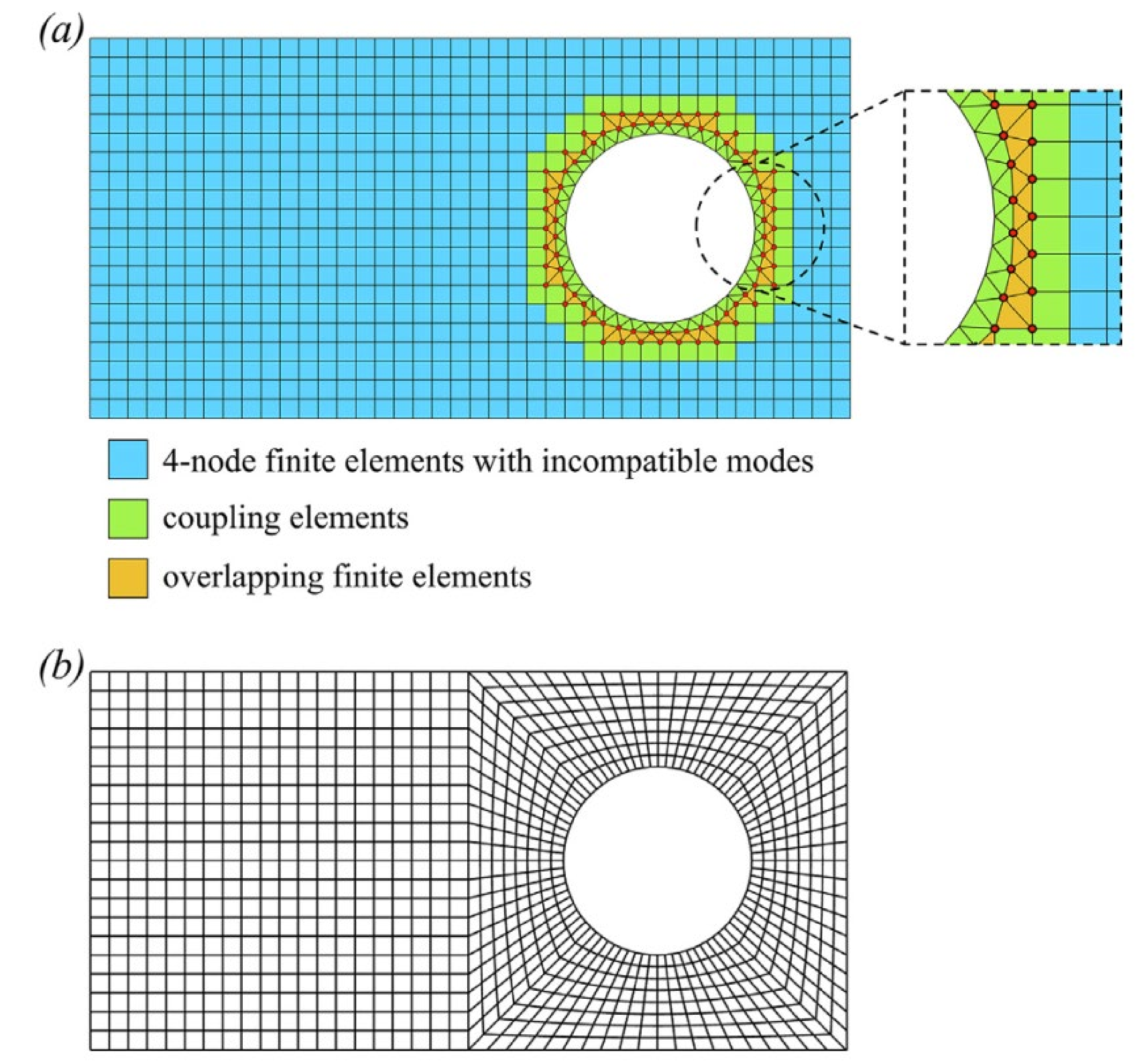
Static analysis of a 2D domain with a hole – the hole could be a tunnel



The meshes used

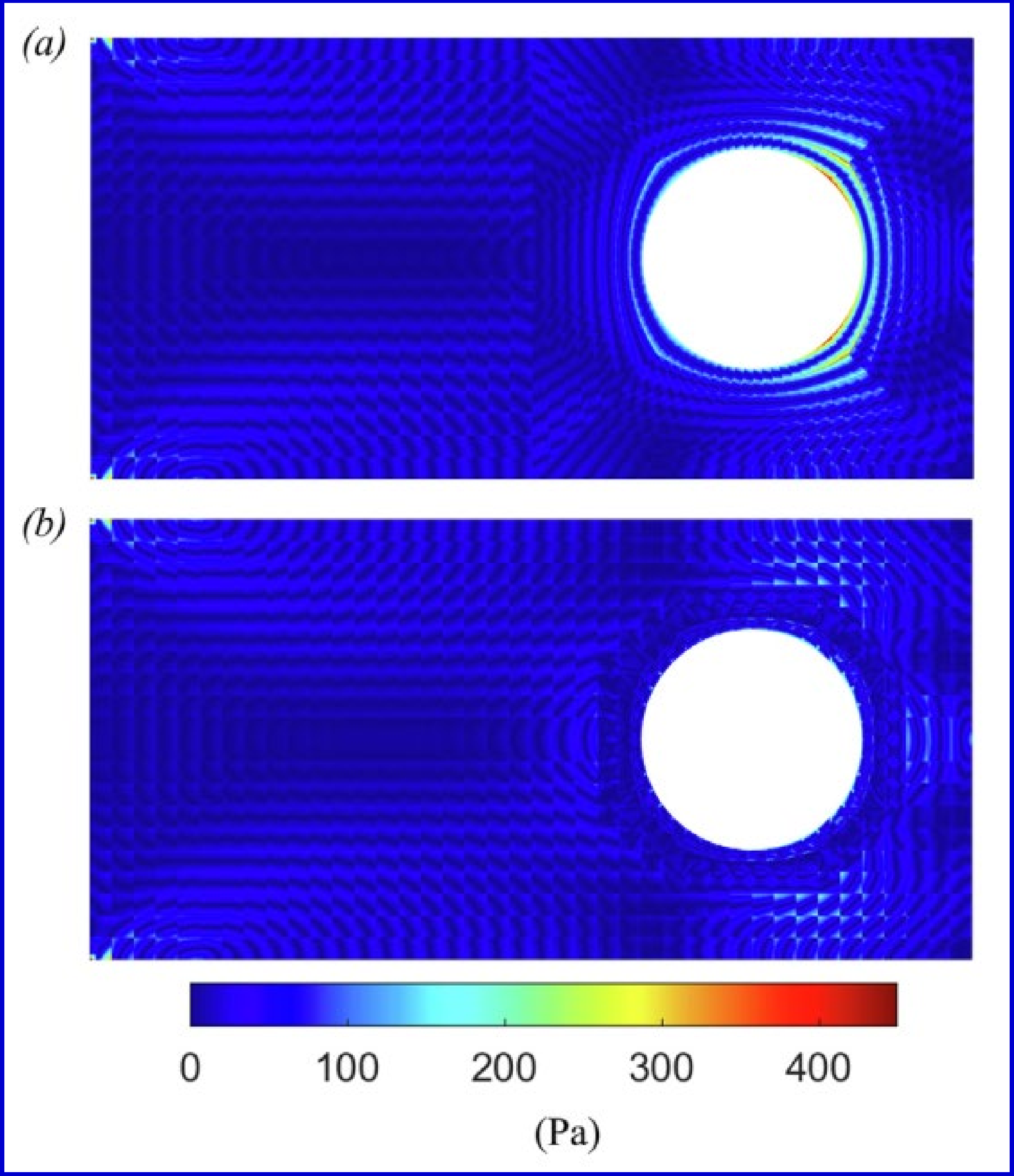
AMORE mesh (2,430 dofs)

4-node ICM mesh (2,238 dofs)



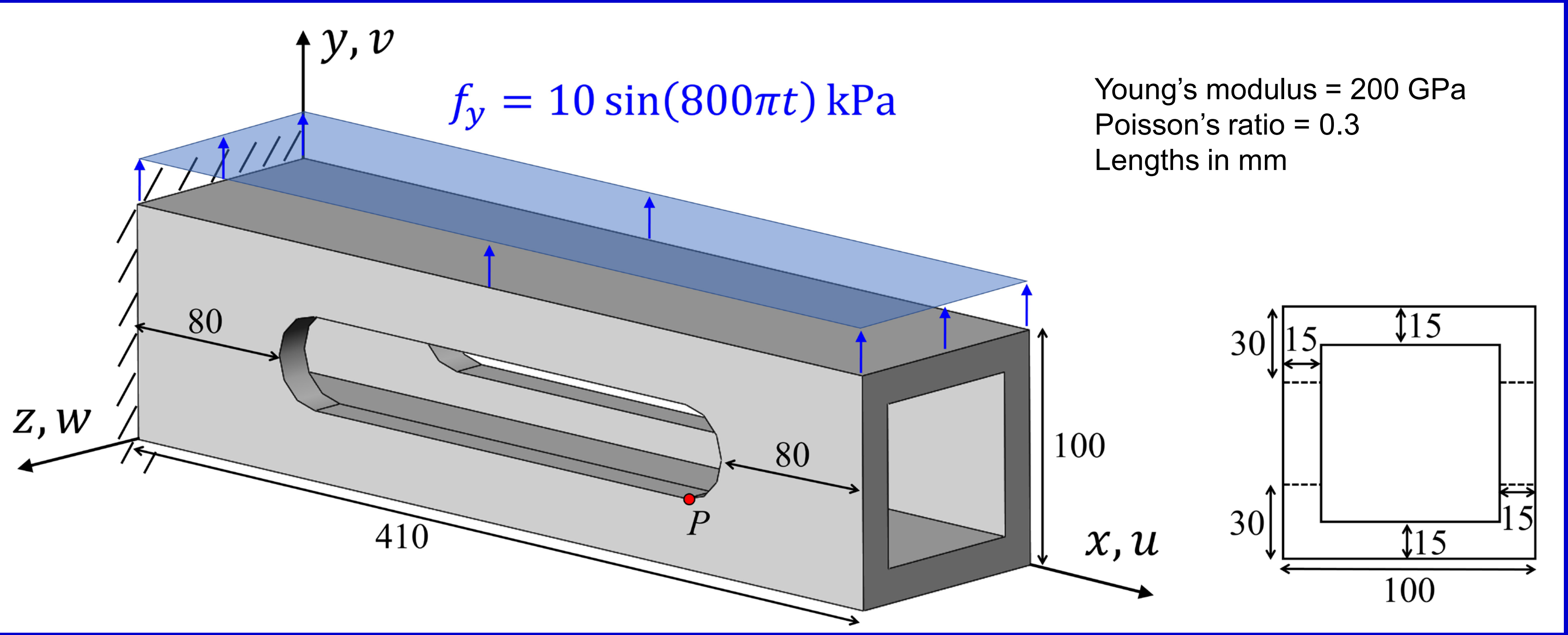
4-node ICM mesh

AMORE mesh



Absolute error in von Mises stress (max. von Mises stress = 2,400 Pa)

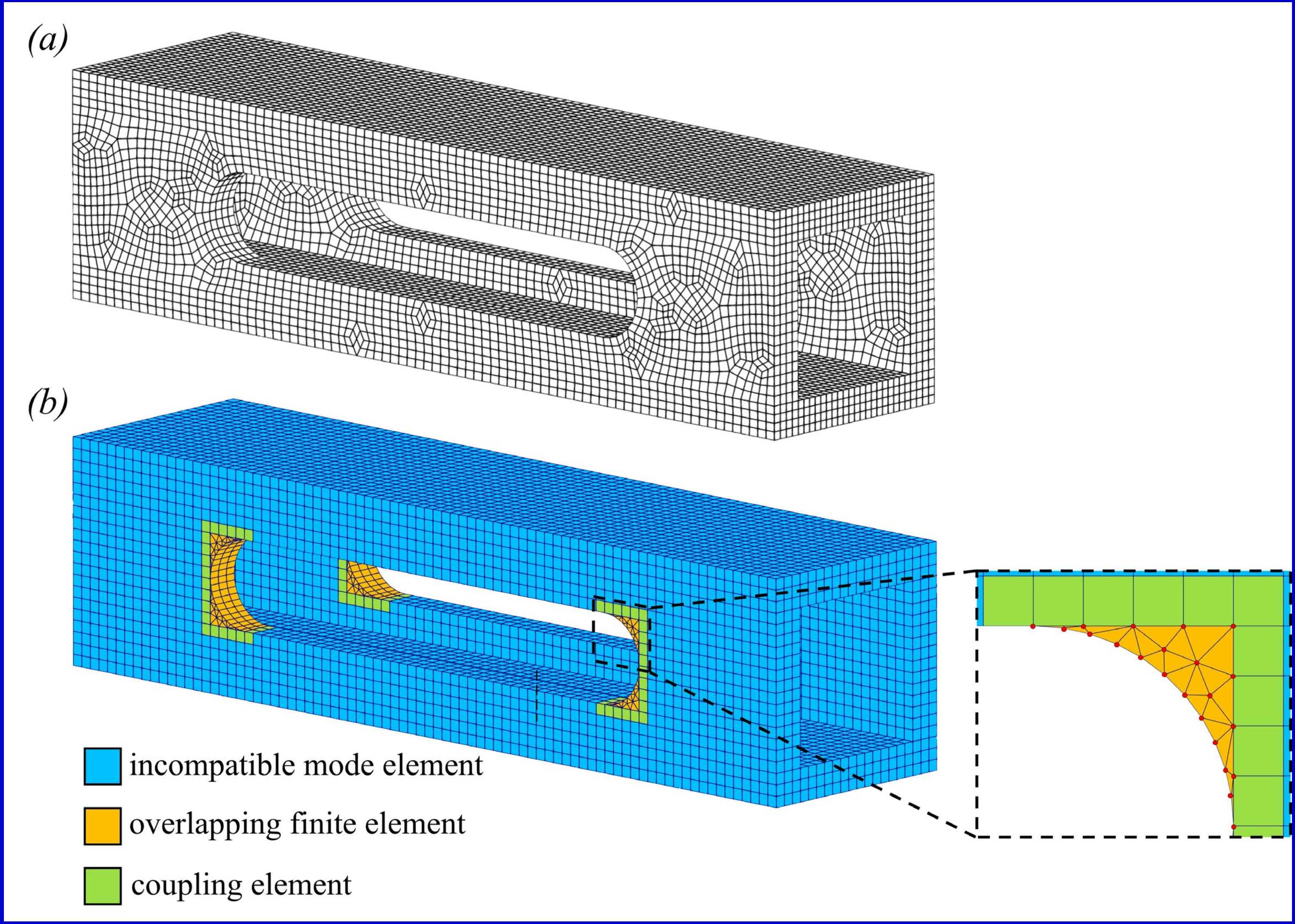
Mode superposition analysis of 3D bracket



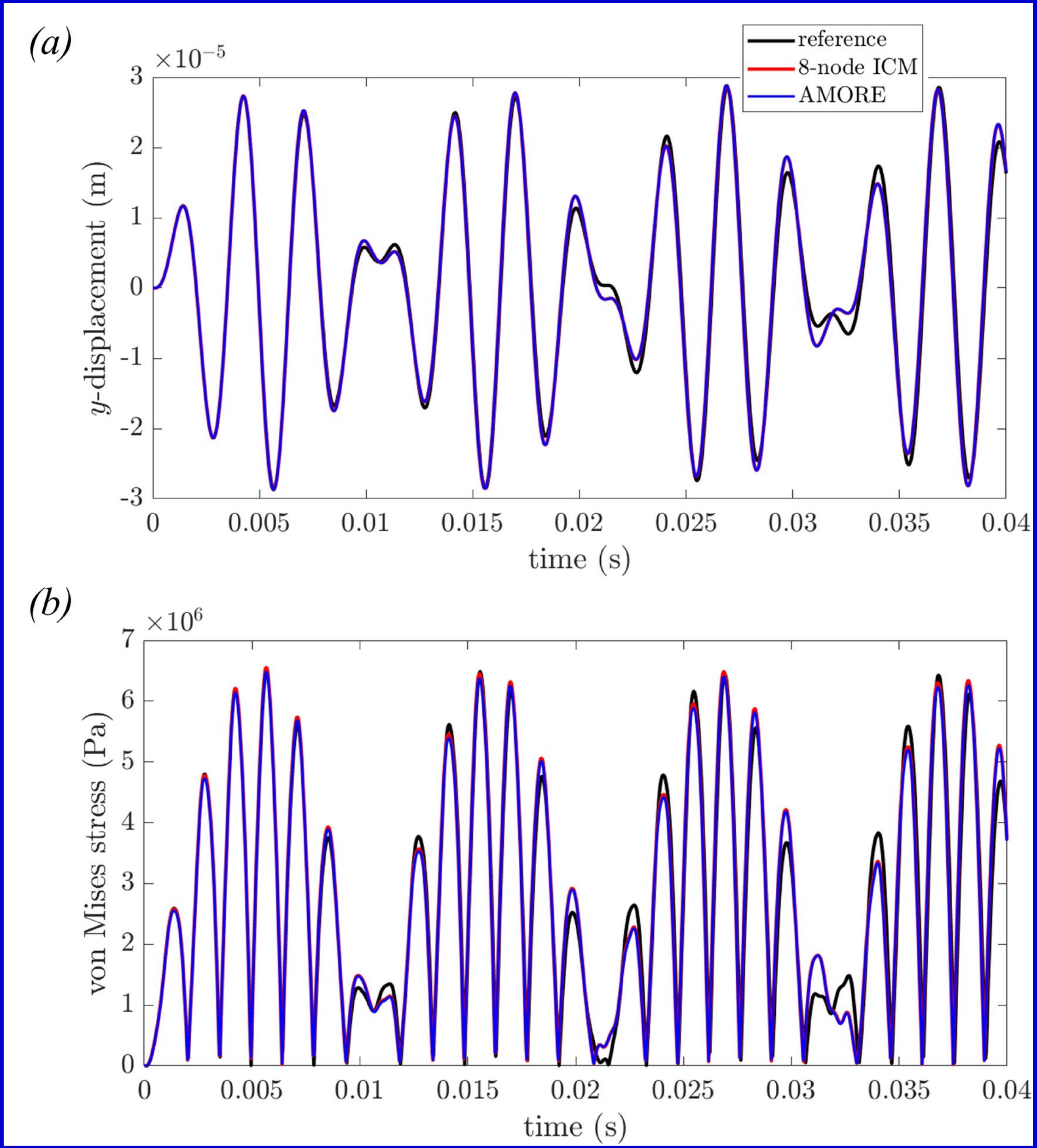
The meshes used

8-node ICM mesh (106,866 dofs)

AMORE mesh (66,120 dofs)



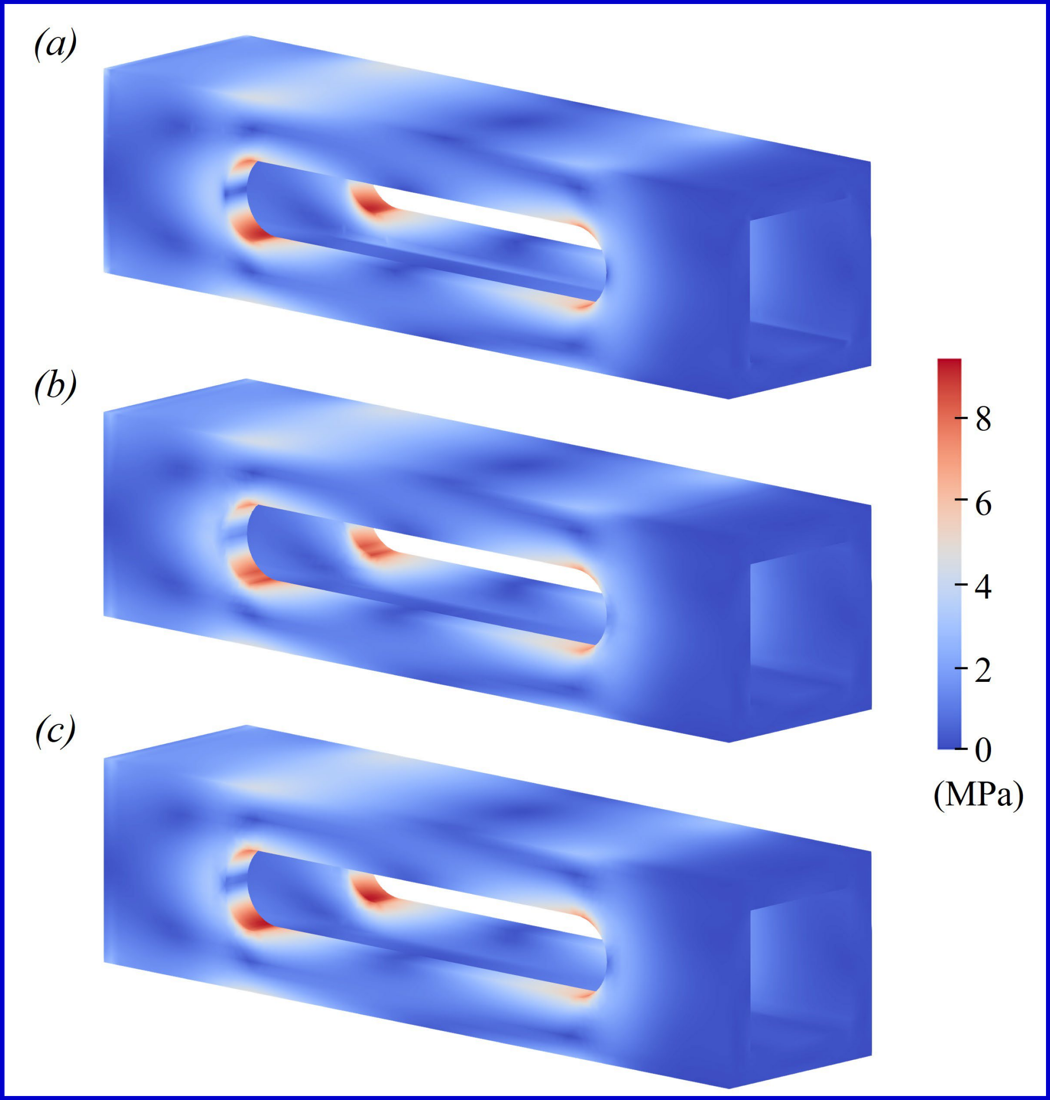
Numerical predictions at point P



Ref.

8-node ICM mesh (106,866 dofs)

AMORE mesh (66,120 dofs)



Von Mises stress predictions at t = 0.027 s

Analysis of wave propagations

Need to have "effective spatial interpolations" and an "effective time integration".

The effective spatial interpolation is given by the use of overlapping elements.

The effective time integration is given by the use of the Bathe implicit time integration scheme.

In the overlapping finite element degrees of freedom we use the bilinear and harmonic functions.

For wave propagation solutions, we assume in 2D analyses:

$$V_I^h = \text{span}\left\{\begin{array}{l} 1, x, y, (xy), \\ \cos\left(\dfrac{2\pi k_x x}{\Lambda_x}\right), \sin\left(\dfrac{2\pi k_x x}{\Lambda_x}\right), \cos\left(\dfrac{2\pi k_y y}{\Lambda_y}\right), \sin\left(\dfrac{2\pi k_y y}{\Lambda_y}\right) \\ \cos\left(\dfrac{2\pi k_x x}{\Lambda_x} \pm \dfrac{2\pi k_y y}{\Lambda_y}\right), \sin\left(\dfrac{2\pi k_x x}{\Lambda_x} \pm \dfrac{2\pi k_y y}{\Lambda_y}\right) \end{array}\right\}$$

$$k_x, k_y = 1, \ldots, p \quad with \ p \in \{1, 2, 3\}$$

The Bathe implicit time integration scheme uses 2 sub-steps per time step, with splitting ratio γ

Assume $\gamma = 0.5$; the first sub-step uses

$$\mathbf{M}\,{}^{t+\Delta t/2}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t/2}\dot{\mathbf{U}} = {}^{t+\Delta t/2}\mathbf{R} - {}^{t+\Delta t/2}\mathbf{F}$$

$${}^{t+\Delta t/2}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{\Delta t}{4}\left({}^{t}\ddot{\mathbf{U}} + {}^{t+\Delta t/2}\ddot{\mathbf{U}}\right)$$

$${}^{t+\Delta t/2}\mathbf{U} = {}^{t}\mathbf{U} + \frac{\Delta t}{4}\left({}^{t}\dot{\mathbf{U}} + {}^{t+\Delta t/2}\dot{\mathbf{U}}\right)$$

The second sub-step uses:

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}$$

$${}^{t+\Delta t}\dot{\mathbf{U}} = \frac{1}{\Delta t}\,{}^{t}\mathbf{U} - \frac{4}{\Delta t}\,{}^{t+\Delta t/2}\mathbf{U} + \frac{3}{\Delta t}\,{}^{t+\Delta t}\mathbf{U}$$

$${}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{1}{\Delta t}\,{}^{t}\dot{\mathbf{U}} - \frac{4}{\Delta t}\,{}^{t+\Delta t/2}\dot{\mathbf{U}} + \frac{3}{\Delta t}\,{}^{t+\Delta t}\dot{\mathbf{U}}$$

An important point is that we can vary the time step splitting ratio γ, to then use the ρ_∞ - Bathe method

We use – for the 1st sub-step

$$^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \left(\frac{\gamma\Delta t}{2}\right)\left({}^{t}\dot{\mathbf{U}} + {}^{t+\gamma\Delta t}\dot{\mathbf{U}}\right)$$

$$^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \left(\frac{\gamma\Delta t}{2}\right)\left({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right)$$

and for the 2nd sub-step

$$^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \Delta t\ (q_0\,{}^{t}\dot{\mathbf{U}} + q_1\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + q_2\,{}^{t+\Delta t}\dot{\mathbf{U}})$$

$$^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \Delta t\ (q_0\,{}^{t}\ddot{\mathbf{U}} + q_1\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + q_2\,{}^{t+\Delta t}\ddot{\mathbf{U}})$$

There are two natural parameters to work with: the splitting ratio γ and the spectral radius ρ_∞, giving the ρ_∞ - Bathe scheme

For second-order accuracy we use:

$$q_0 = (\gamma - 1) q_1 + \frac{1}{2}; \quad q_2 = -\gamma q_1 + \frac{1}{2}$$

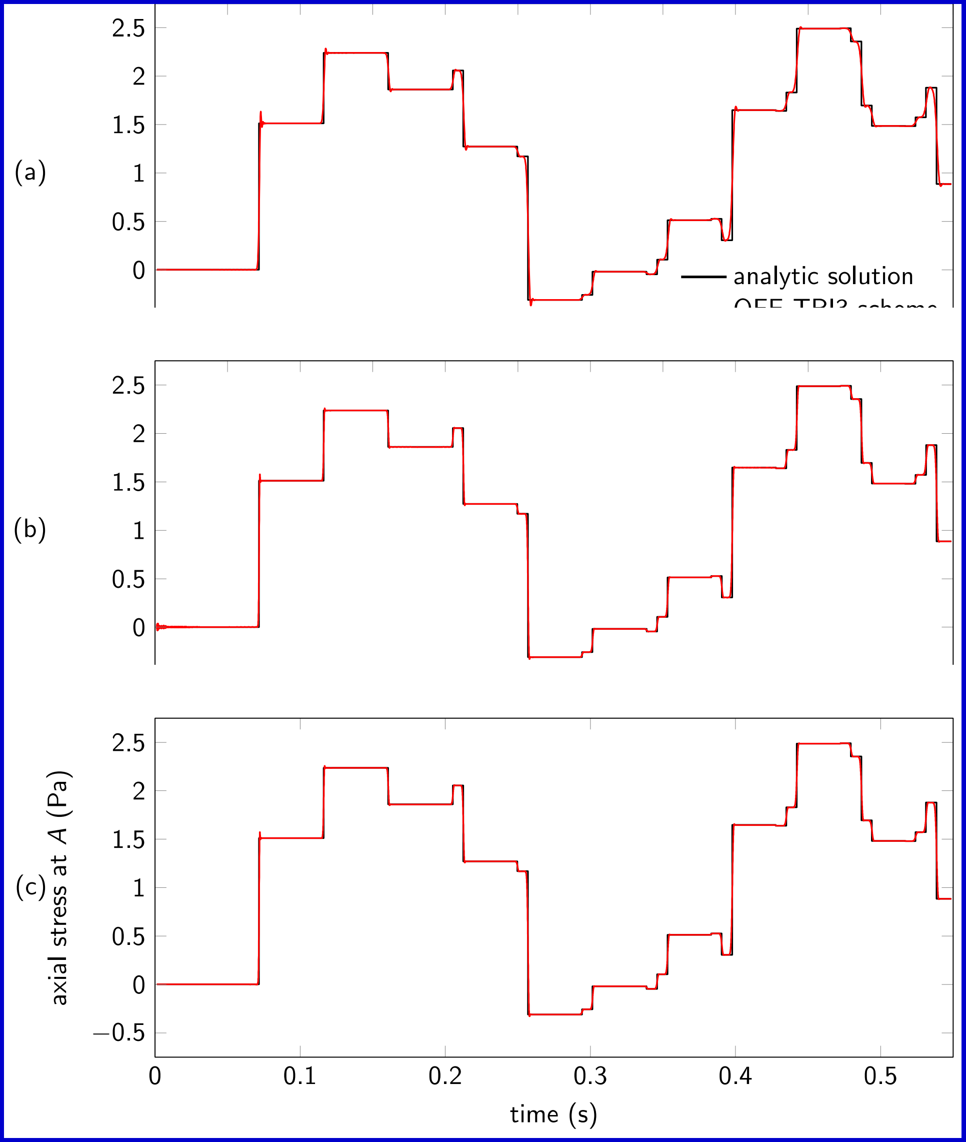
$$q_1 = \frac{\rho_\infty + 1}{2\gamma(\rho_\infty - 1) + 4} \qquad \rho_\infty \in [0,1]$$

For wave propagation analyses we use:

$$q_0 = q_1 = 0.325 \qquad q_2 = 0.35 \qquad \gamma = 0.5$$

Some example solutions

1D analysis of bar, subjected to step loading, 160 4-node overlapping finite elements, $\gamma = 0.5$

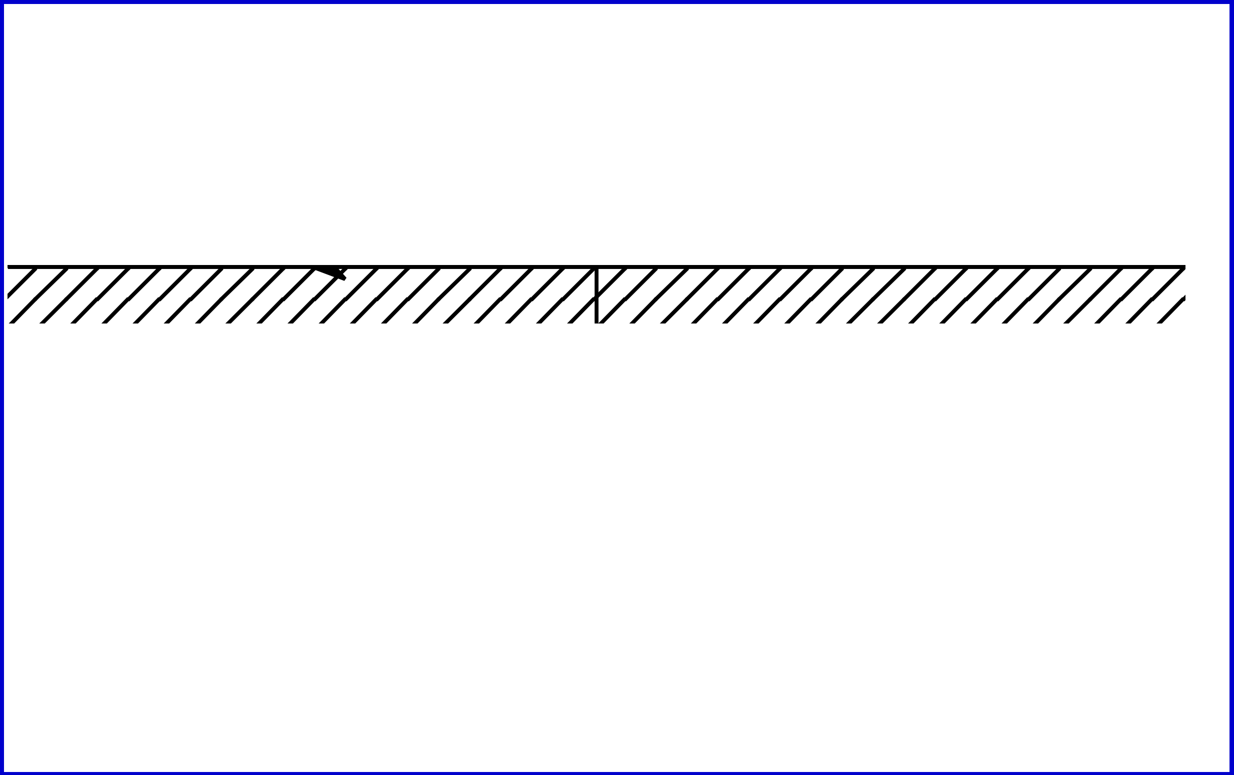


Use of uniform mesh and c_2

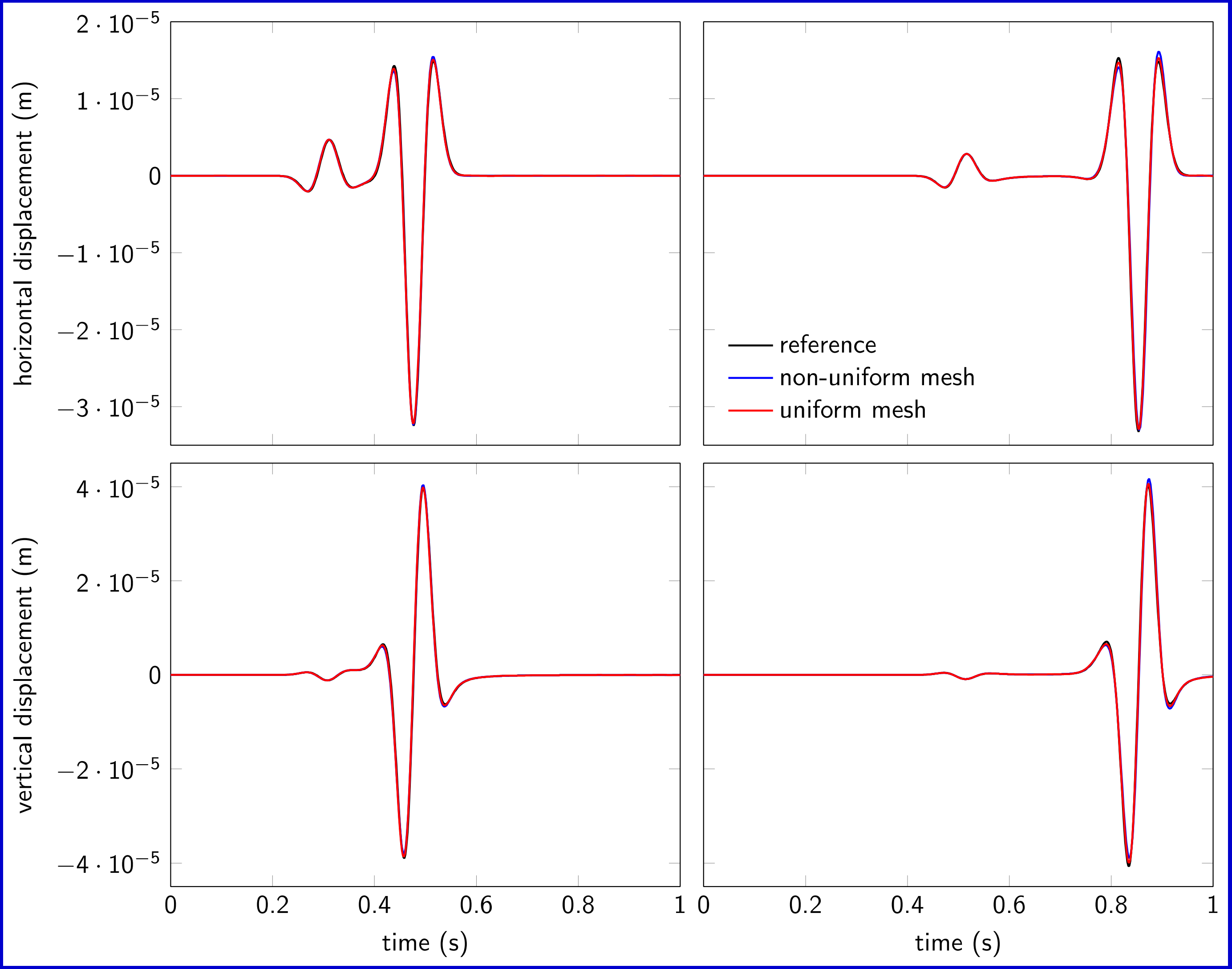
Use of uniform mesh and c_1

Use of non-uniform mesh and c_1

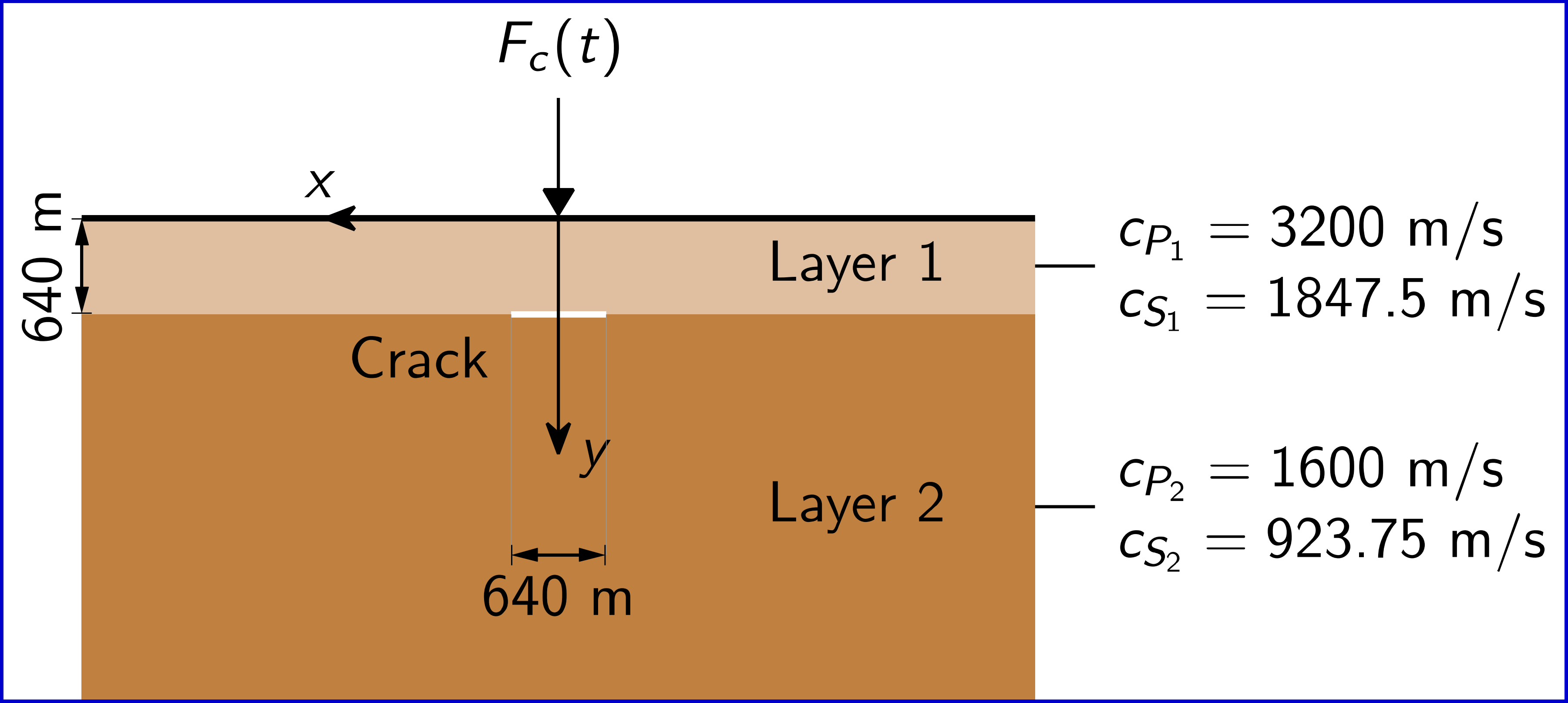
Axial stress at point A in bar as a function of time, time step selected by different wave speeds, CFL = 0.125



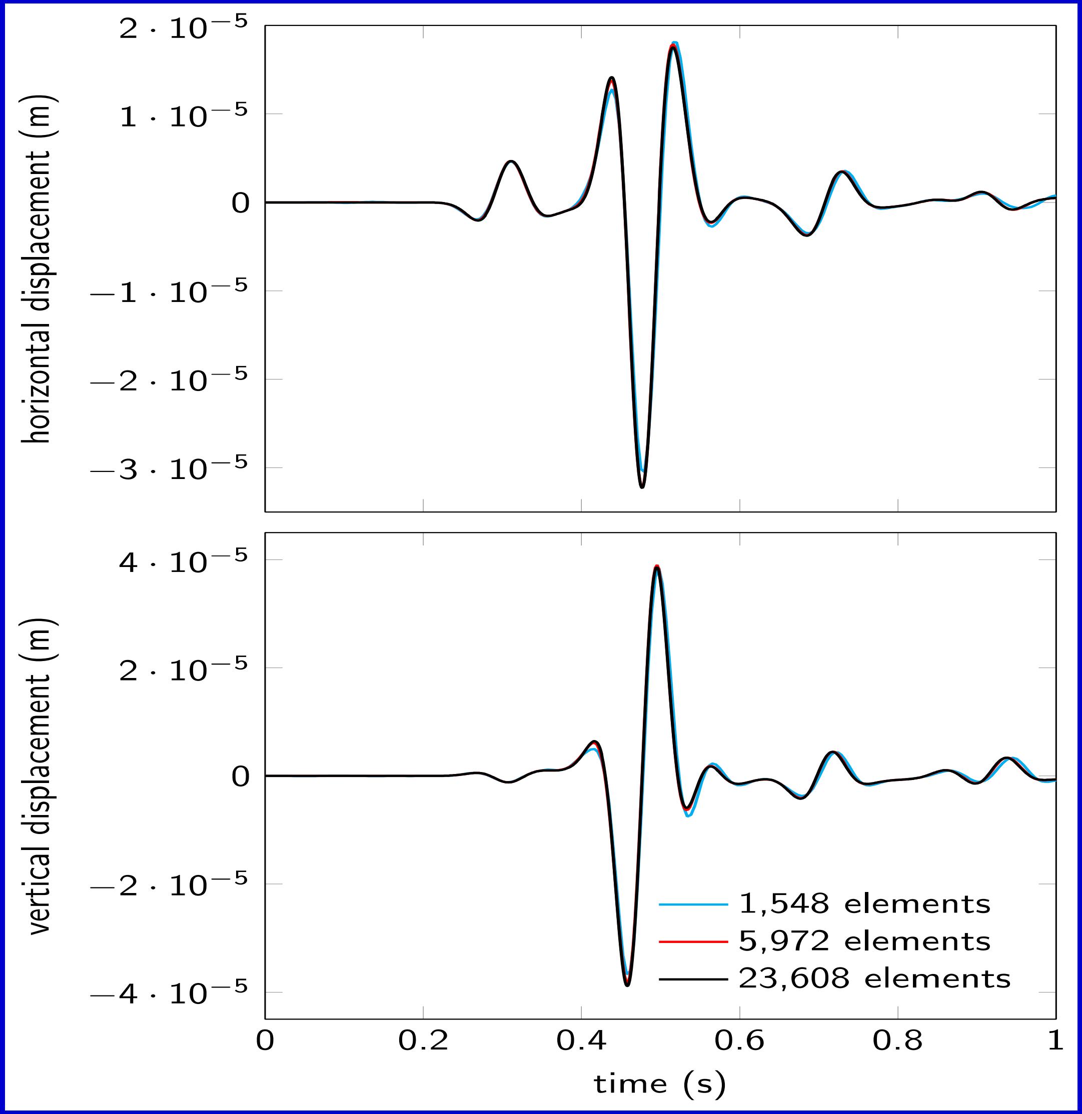
Analysis of semi-infinite elastic medium subjected to a concentrated load of Ricker wavelet



Response at x = (640,0) on left and x = (1280, 0) on right; uniform mesh of 8,192 elements & non-uniform mesh of 5,000 elements, symmetry of analysis domain is used



Medium with crack subjected to concentrated Ricker wavelet force



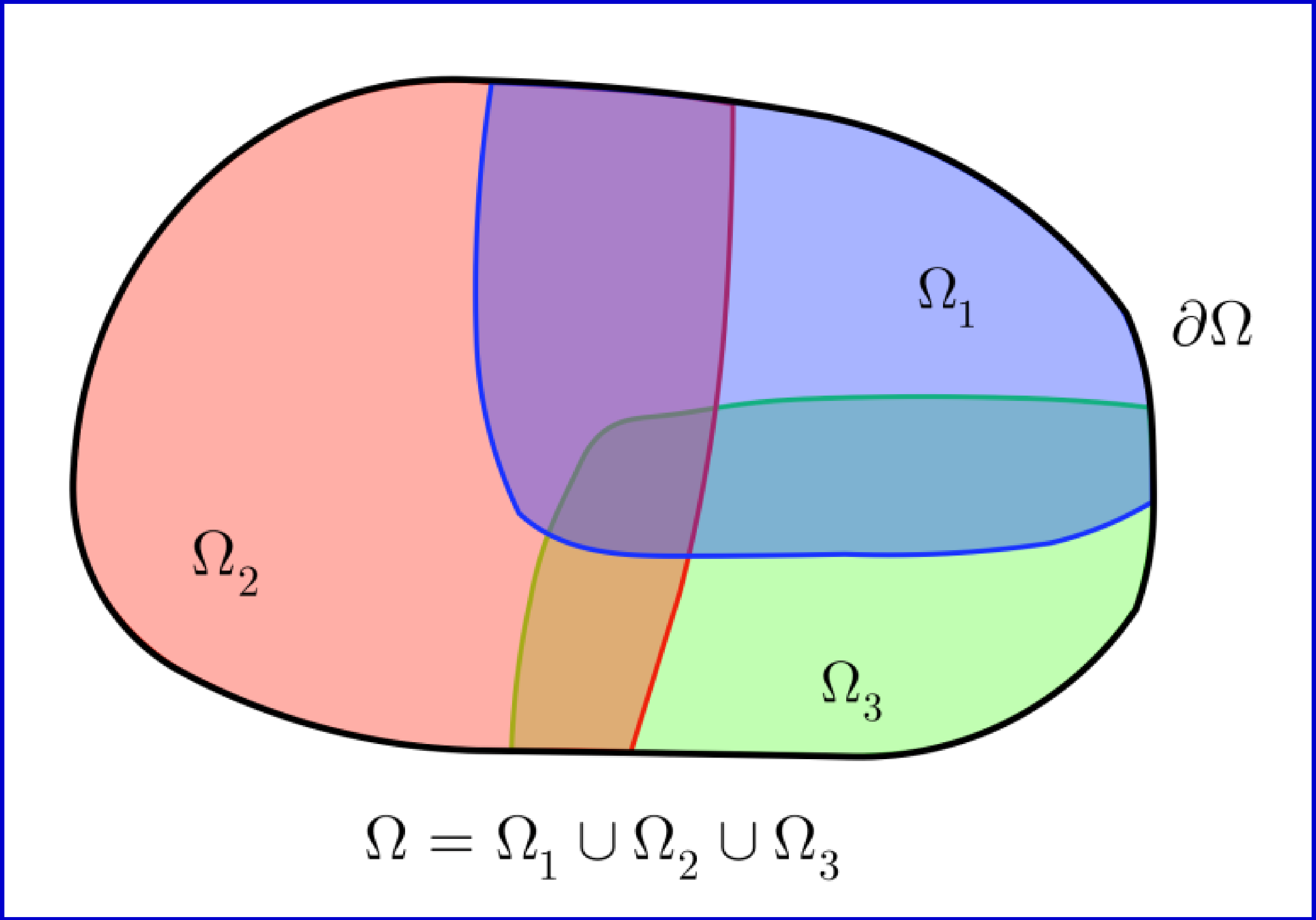
Calculated response as mesh is refined, at x= (640,0)

Only Briefly ---

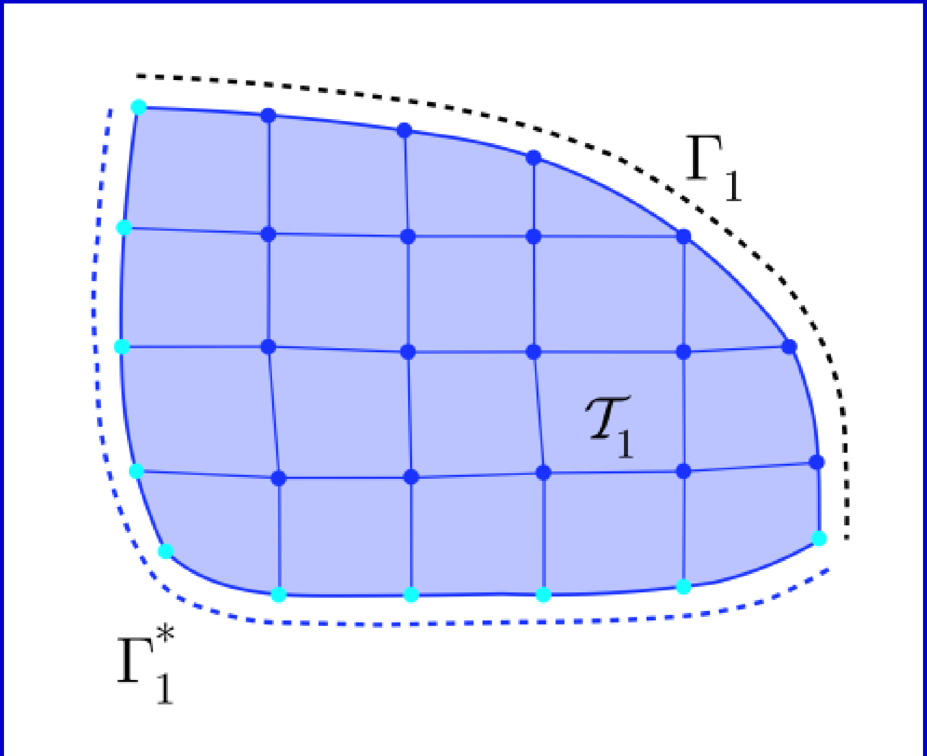
AMORE with overlapping meshes

We can also use ‘overlapping meshes’

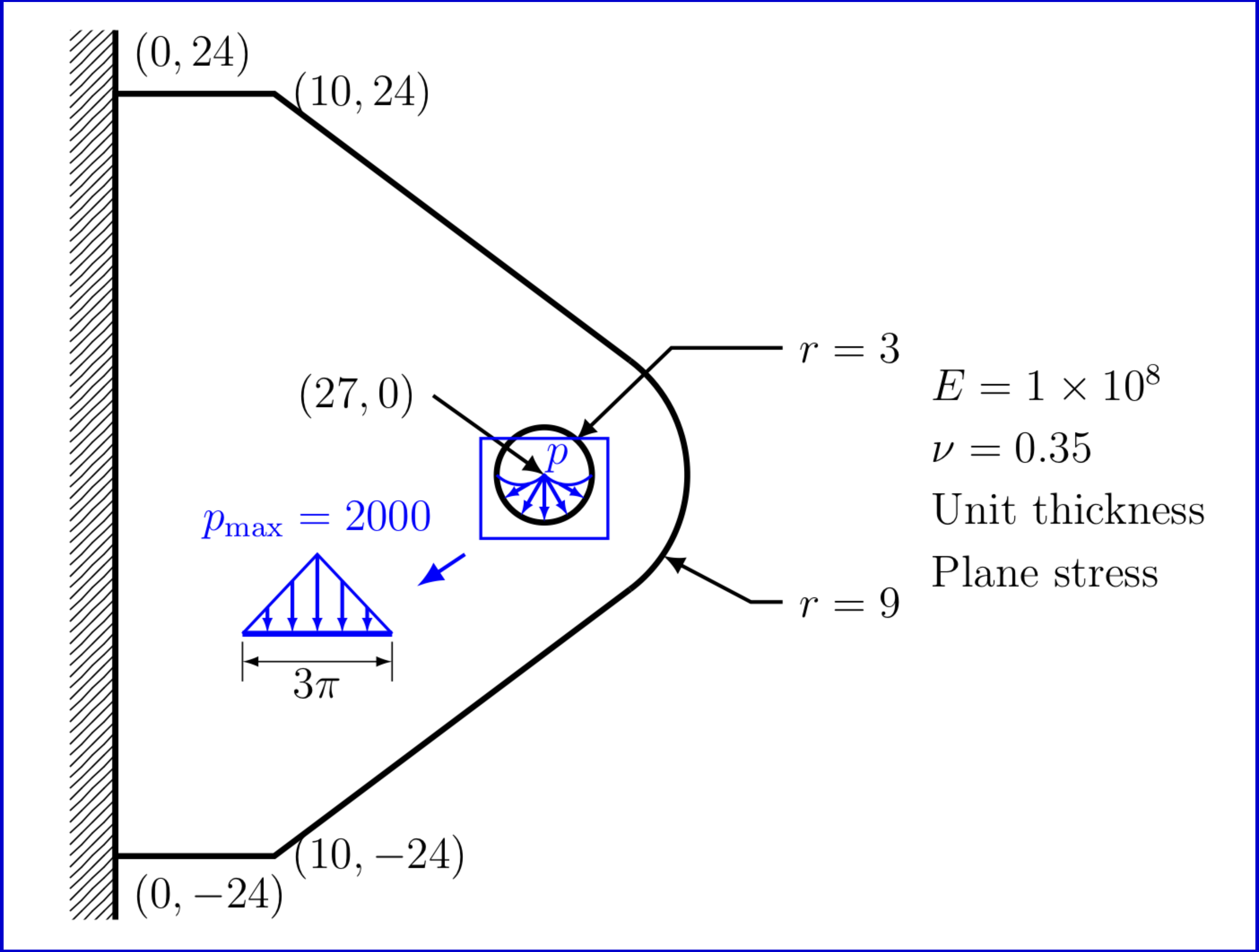
Here we consider the overlapping
of meshes of regular elements, but the
meshes could in principle also contain
overlapping elements



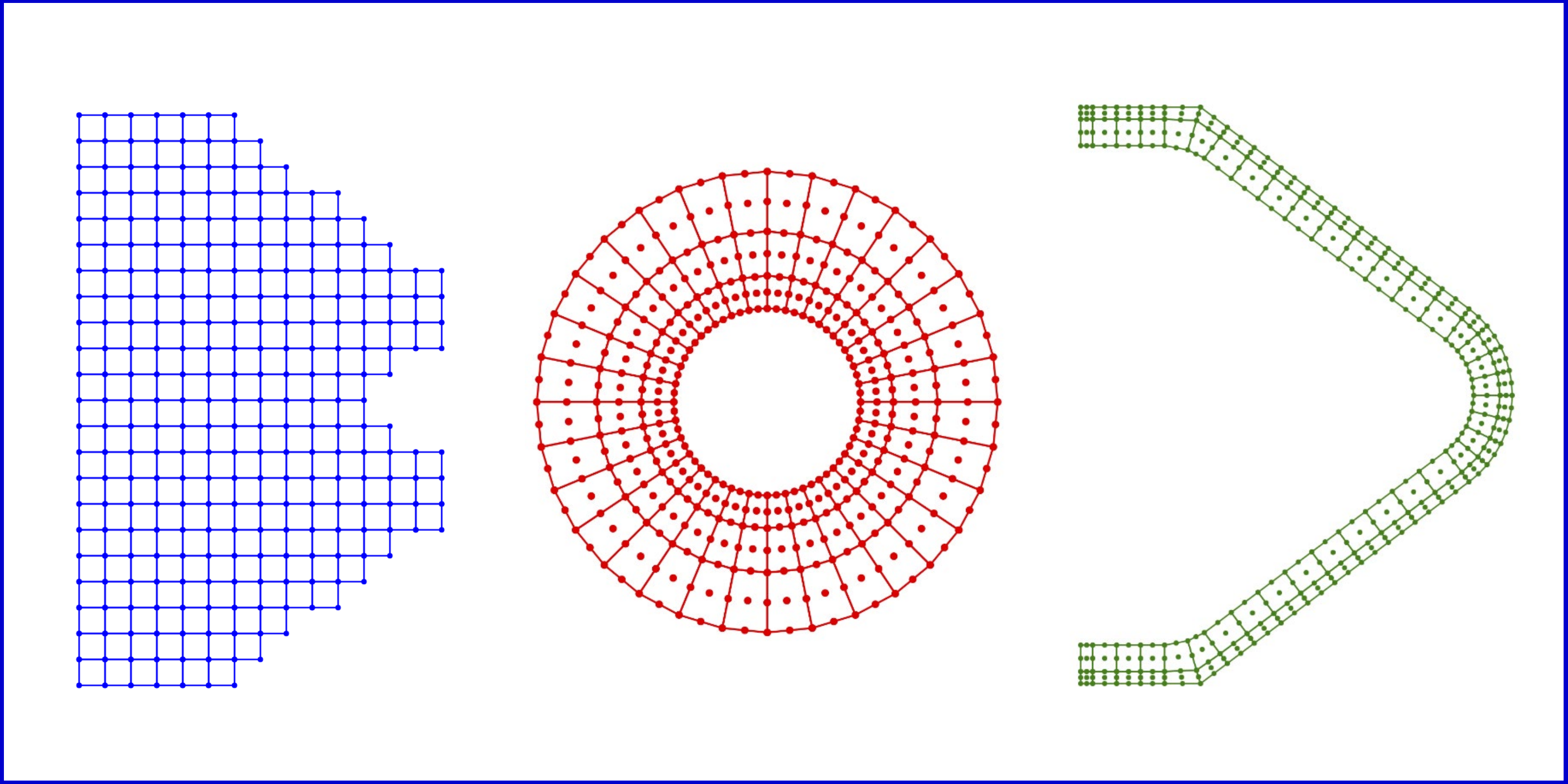
Three overlapping regions



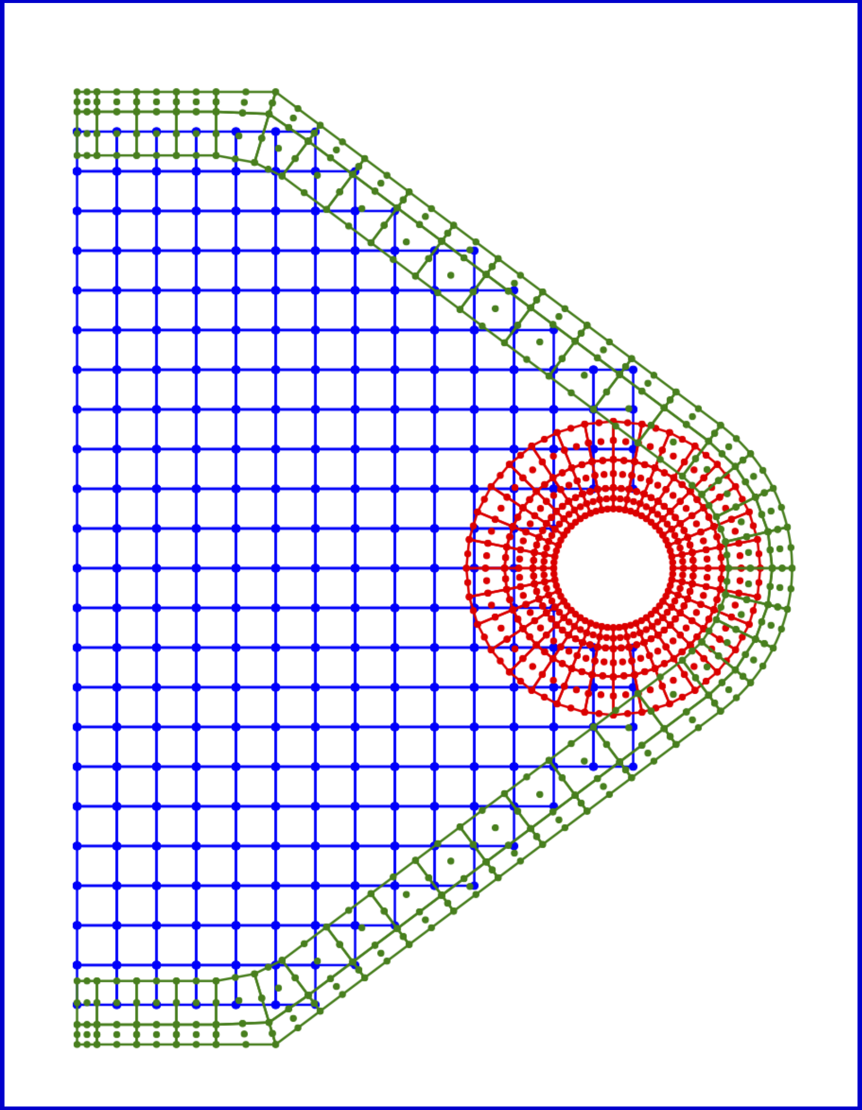
Regions are discretised by traditional 4-node elements; overlapping elements could be used



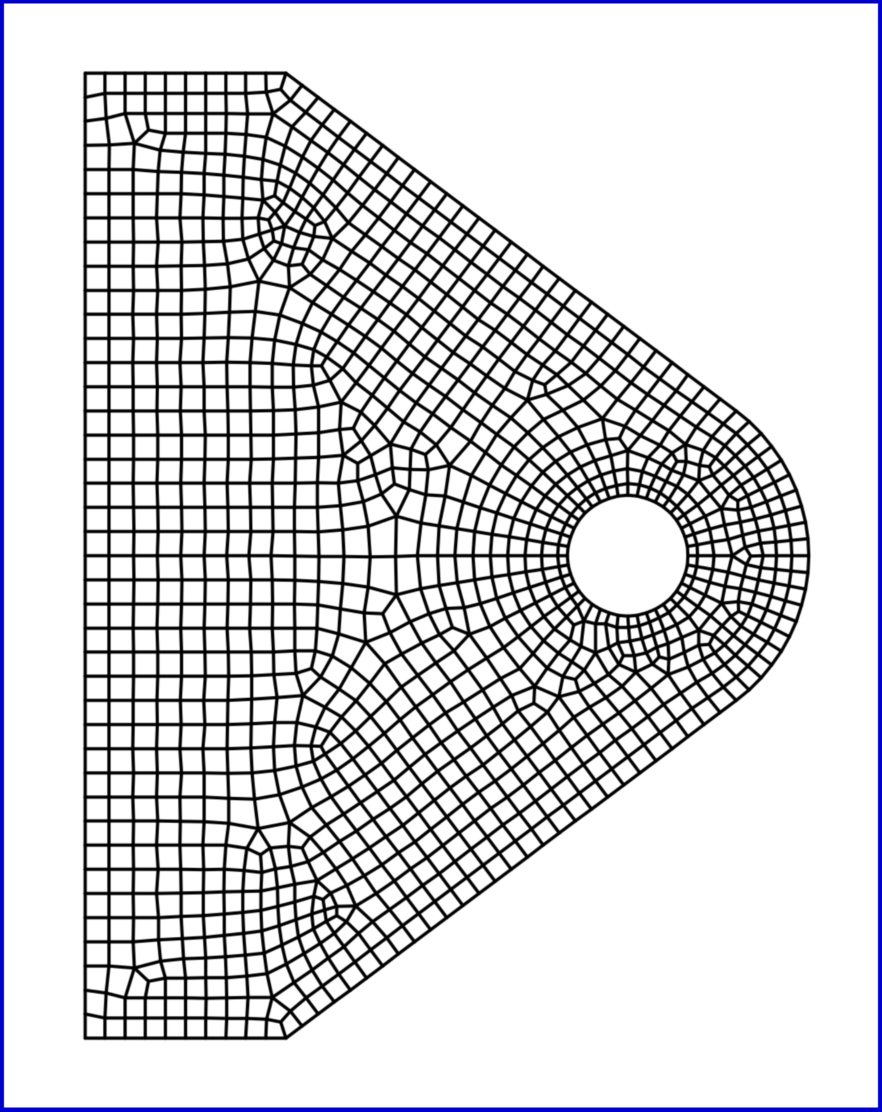
Analysis of a bracket



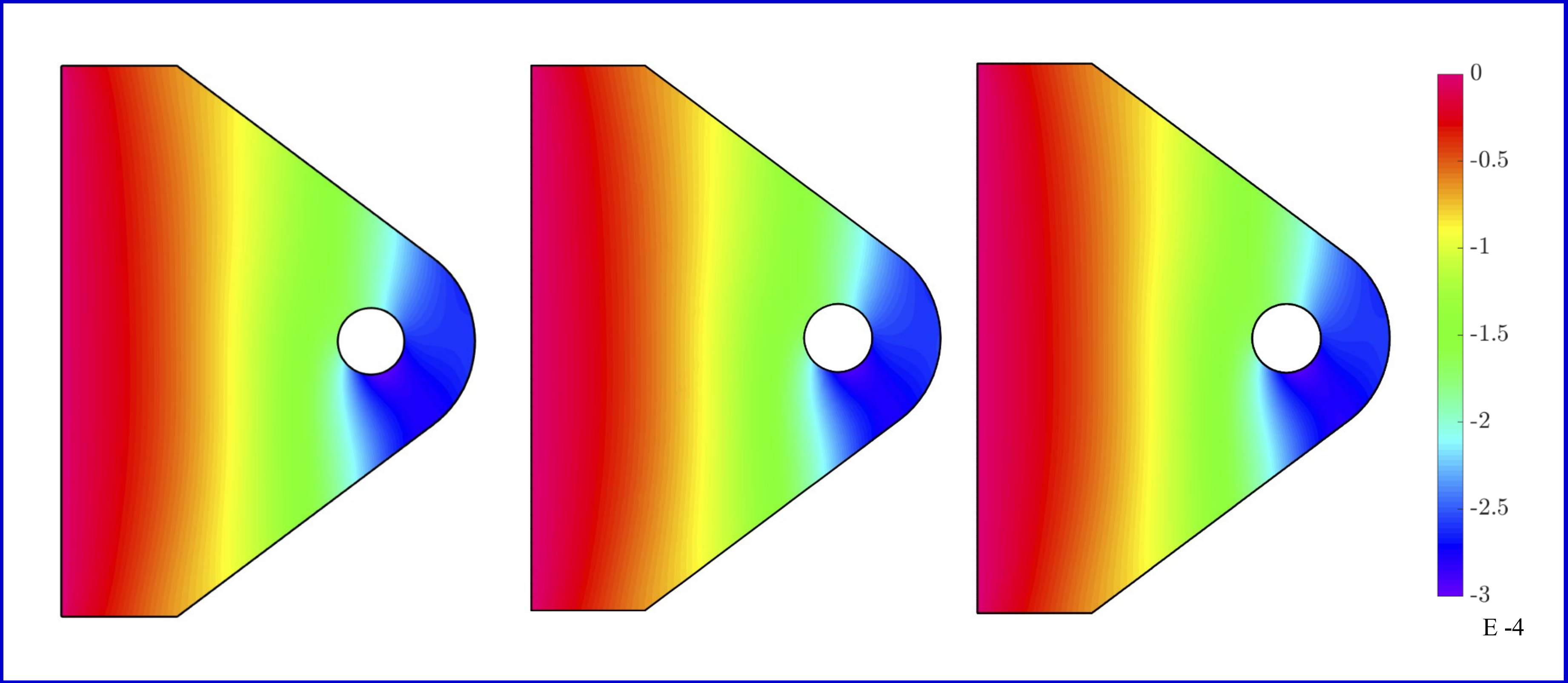
The three meshes used



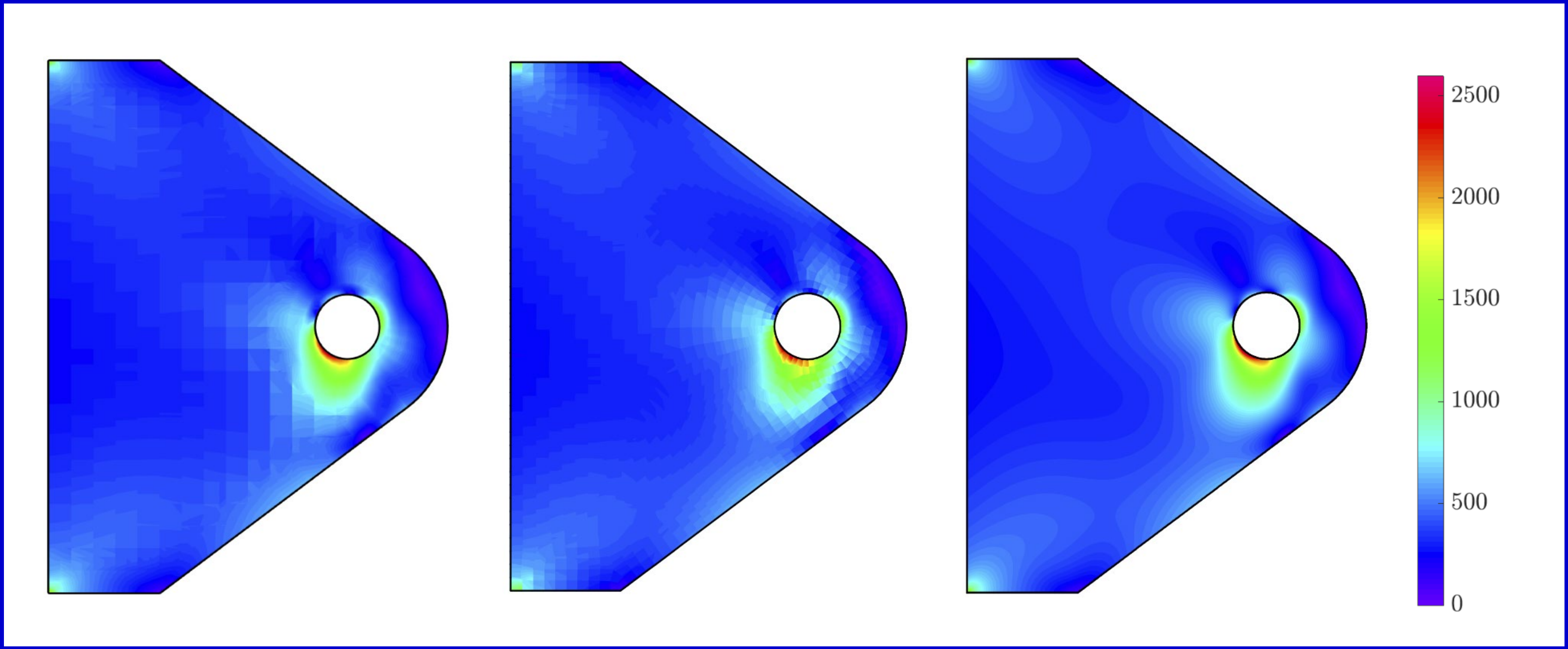
The 3 meshes to represent the analysis domain



The traditional mesh used
for comparison of analysis results



Solutions of vertical displacement; left: AMORE; middle: Finite element solution; right: Reference



von Mises stress predictions

However, we will in this presentation focus on the 'overlapping finite elements'

Concluding remarks

The AMORE paradigm of performing finite element analyses is general and can be effective to reduce time in meshing and computations, giving accurate solutions and thus increases reliability of analyses.

An important ingredient in AMORE are the “Overlapping Finite Elements (OFE)”.

The OFE are a generalization of the “traditional (regular) finite elements” and “finite elements with interpolation covers”.

The OFE are of much value in AMORE and also – in general -- when the solution to be obtained is sensitive to element distortions

We formulated the OFE and focused on the "fundamental performance"of the elements, some computational cost comparisons and on the use of AMORE

In the analyses of wave propagations, we obtained very accurate solutions; here traditional finite elements can hardly be used for accurate response predictions

However –

The effective use of OFE requires the use of special meshing schemes to take full advantage of the use of undistorted (regular) elements in most of the analysis domain.

The regular elements would typically use incompatible modes which give good predictive capability when the elements are undistorted.

Good applications of the AMORE scheme can be found in the --

- analyses of 3D tunnels, without and with internal linings;
- 3D safety analyses of mines;
- analyses of dams;
- simulation of 3D sructures in structural health monitoring;
- 3D analyses of thick parts of structures, e. g. connections.

In Conclusion: The use of AMORE and OFE will open new avenues for computational models & finite element analyses.

And further research can be pursued:

-- Fundamental improvements in the OFE can be looked for.

-- When applied to solve large displacement, large strain, multi-physics problems, the insensitivity of the OFE to element distortions will be particulary valuable.

References:

KJ Bathe, Finite Element Analyses – *En Plus.* Springer Verlag, 2025, in press.

In this book, many papers are cited that give detailed information on the OFE procedures, time integration schemes, and other finite element procedures (including the use of Machine Learning)

Further fundamental information is given in:

KJ Bathe, Finite Element Procedures, 2nd edn KJ Bathe amazon.com, Springer-Verlag 2025, in press.